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MESSENGER OF MATHEMATICS.

NOTES ON THE THEORY OF ELLIPTIC TRANSFORMATION.

By the late Professor *H. J. S. Smith.*

[The following pages contain the fragments, relating to the continuation of the Notes published in pp. 49–99 of the last volume, which were found among Prof. Smith's MSS. It is to be understood that they are only unfinished work which he would have greatly altered and extended. The Notes have been placed in what appears to be the most convenient order, and headings have been supplied.]

III.

On the Functions $Q(\omega)$ and $Q'(\omega)$.

LET $Q(\omega)$, $Q'(\omega)$ be two functions of ω defined by the equations

$$(1)\quad \begin{cases} iQ(\omega) = -\tfrac{1}{2}\pi \dfrac{d}{d\omega}\left(\dfrac{1}{K}\right) = \dfrac{\pi}{2K^2}\dfrac{dK}{d\omega}, \\ iQ'(\omega) = \tfrac{1}{2}\pi \dfrac{d}{d\left(-\frac{1}{\omega}\right)}\left(\dfrac{1}{K'}\right) = \dfrac{\pi}{2K^2}\dfrac{dK'}{d\omega}, \end{cases}$$

which imply the equality

$$(2)\qquad Q'(\omega) = -Q\left(-\frac{1}{\omega}\right),$$

because $K'(\omega) = K\left(-\dfrac{1}{\omega}\right)$; and which may be replaced by either of the following pairs:

$$(3)\qquad Q = 2k^2k'^2\frac{dK}{d.k^2}, \qquad Q' = 2k^2k'^2\frac{dK'}{d.k^2},$$

$$(4)\qquad Q = k^2K - J, \qquad Q' = k^2K' - J',$$

of which (3) was obtained by writing in (1) for $d\omega$ its value $-\frac{i\pi d.k^2}{4k^2k'^2K^2}$, and (4) by substituting in (3) the values of $\frac{dK}{d.k^2}$ and $\frac{dK'}{d.k^2}$ taken from the equations (i) and (v) of Art. 12.*

The values of $\frac{dJ}{d.k^2}$ and $\frac{dJ'}{d.k^2}$ taken from the same equations give

$$(5) \qquad \frac{dQ}{d.k^2} - \tfrac{1}{2}K = 0, \qquad \frac{dQ'}{d.k^2} - \tfrac{1}{2}K' = 0,$$

and eliminating between these equations and the equation (3), we have, [writing x for k^2],

$$(6) \qquad \frac{d^2Q}{dx^2} = \frac{Q}{4x(1-x)},$$

the complete solution being $CQ + C'Q'$.

The equations (1) and (5) respectively give

$$(7) \qquad \frac{Q'}{Q} = \frac{dK'}{dK}; \qquad \frac{dQ'}{dQ} = \frac{K'}{K},$$

of which either is a consequence of the other, because

$$(8) \qquad K'Q - KQ' = KJ' - K'J = \tfrac{1}{2}\pi.$$

We have also

$$\frac{1}{K}\frac{dQ}{d\omega} = \frac{1}{K'}\frac{dQ'}{d\omega} = -\frac{2k^2k'^2K^2}{i\pi} = \tfrac{1}{2}\frac{d.k^2}{d\omega},$$

$$\frac{1}{Q}\frac{dK}{d\omega} = \frac{1}{Q'}\frac{dK'}{d\omega} = -\frac{2}{i\pi}K^2,$$

$$\frac{Q'dQ - QdQ'}{d\omega} = \tfrac{1}{4}\pi\frac{d.k^2}{d\omega}.$$

Lastly, combining the equations (4) with the equations (vi) of Art. 10, we have

$$(9) \qquad \begin{aligned} Q &= \int_0^K k^2\cos^2\text{am}\,x\,dx, \\ Q' &= -i\int_K^{K+iK'} k^2\cos^2\text{am}\,x\,dx; \end{aligned}$$

* These references are to his Memoir on the Theta and Omega Functions, which will appear in the same volume as the Tables of the Theta Functions.

thus the functions Q and Q' do not differ essentially from the functions $J = k^2K - Q$, $J' = k^2K' - Q'$ of M. Weierstrass, nor from the functions

$$E = K - J = k'^2K + Q, \quad E' = J' = k^2K' - Q'$$

of Legendre: their introduction serves to simplify the formula relating to the transformation of the complete functions of the second species.

If $\omega = \begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix} \times \Omega$ is any transformation whatever of determinant n, that is, if $\omega = \dfrac{\gamma + \delta\Omega}{\alpha + \beta\Omega}$, we have

$$\frac{K(\omega)}{M} = \alpha K(\Omega) + i\beta K'(\Omega),$$

$$i\frac{K'(\omega)}{M} = \gamma K(\Omega) + i\delta K'(\Omega).$$

Differentiating with regard to k^2, multiplying by $-2k^2k'^2M$, and observing that

$$d.\lambda^2 = \frac{1}{nM^2} \times \frac{\lambda^2\lambda'^2}{k^2k'^2}\, d.k^2,$$

$$Q(\Omega) = 2\lambda^2\lambda'^2\frac{d.K(\Omega)}{d.\lambda^2}, \qquad Q'(\Omega) = 2\lambda^2\lambda'^2\frac{d.K'(\Omega)}{d.\lambda^2},$$

we find

$$\frac{1}{nM}[\alpha Q(\Omega) + i\beta Q'(\Omega)] = Q(\omega) - 2k^2k'^2 \; K(\omega)\frac{d.\log M}{d.k^2},$$

$$\frac{1}{nM}[\gamma Q(\Omega) + i\delta Q'(\Omega)] = i\,Q'(\omega) - 2k^2k'^2 iK'(\omega)\frac{d.\log M}{d.k^2}.$$

In these formulæ, which serve to express the transformed functions $Q(\Omega)$, $Q'(\Omega)$ in terms of the given functions $Q(\omega)$, $Q'(\omega)$, $K(\omega)$, $K'(\omega)$, the differential coefficient $\dfrac{1}{M}\dfrac{dM}{d.k^2}$ is to be found from the equation of the Multiplier.

For example, if $\omega = \begin{vmatrix} 3, & 0 \\ 0, & 1 \end{vmatrix} \times \Omega, = \frac{1}{3}\Omega$, then from the equation

$$\frac{1}{M^4} - \frac{6}{M^2} + \frac{8(1-2k^2)}{M} - 3 = 0,$$

of the multiplier, we find without difficulty

$$\frac{dM}{d.k^2} = \frac{-\frac{16}{3}M^4}{(M^2-1)^2}, = \frac{-\frac{4}{3}M^2}{(M-1)^2 + 4k^2M},$$

and the formulæ then become

$$3Q(3\omega) = 3MQ(\omega) + \frac{8k^2k'^2M^2}{(M-1)^2 + 4k^2M} K(\omega),$$

$$3Q'(3\omega) = 3MQ'(\omega) + \frac{8k^2k'^2M^2}{(M-1)^2 + 4k^2M} K'(\omega),$$

where, if ω be a pure imaginary, M is the negative root of the foregoing equation. [Cayley, *On the Transformation of Elliptic Functions*, *Phil. Trans.*, vol. CLXIV. p. 421].

Some considerations of interest present themselves in connexion with the last-mentioned general formulæ.

(1) The two equations are not independent: viz. multiplying the first by $iK'(\omega)$, the second by $K(\omega)$, subtracting and attending to the relations

$$\begin{aligned} Q(\omega)K'(\omega) - Q'(\omega)K(\omega) &= \tfrac{1}{2}\pi, \\ nMK(\Omega) &= \delta K(\omega) - i\beta K'(\omega), \\ inMK'(\Omega) &= -\gamma K(\omega) + i\alpha K'(\omega), \end{aligned}$$

we obtain an identical result.

(2) In the case of Jacobi's first transformation

$$\omega = \begin{vmatrix} n, & 0 \\ 0, & 1 \end{vmatrix} \times \Omega, = \frac{\Omega}{n},$$

the equations become

$$\frac{Q(\Omega)}{M} = Q(\omega) - \frac{2k^2(1-k^2)}{M} K(\omega) \frac{dM}{d.k^2},$$

$$\frac{Q'(\Omega)}{nM} = Q'(\omega) - \frac{2k^2(1-k^2)}{M} K'(\omega) \frac{dM}{d.k^2},$$

of which the first coincides with a relation considered by Prof. Cayley (*Elliptic Functions*, Art. 305), and expressed by him in the form

$$\text{(i)} \quad \frac{1}{nM^2}\left[\lambda'^2 - \frac{G}{\Lambda} - \frac{\lambda\lambda'^2}{M}\frac{dM}{d\lambda}\right] = k'^2 - \frac{E}{K},$$

where

$$\Lambda = K(\Omega), \quad E = \int_0^K \Delta^2 \operatorname{am}(u, k^2)\, du, \quad G = \int_0^\Lambda \Delta^2 \operatorname{am}(u, \lambda^2)\, du,$$

so that $E = k'^2K(\omega) + Q(\omega)$, $G = \lambda'^2K(\Omega) + Q(\Omega)$,

and the equation of Prof. Cayley becomes

$$\text{(ii)}\quad \frac{Q(\omega)}{K(\omega)} = \frac{1}{nM^2}\left[\frac{Q(\Omega)}{K(\Omega)} + \frac{\lambda\lambda'^2}{M}\frac{dM}{d\lambda}\right],$$

which is the same as the first of the foregoing equations, because

$$\frac{K(\omega)}{K(\Omega)} = nM, \quad \frac{\lambda\lambda'^2}{nM^2}\frac{d}{d\lambda} = kk'^2\frac{d}{dk}.$$

If instead of the first transformation we consider any reduced transformation of the type $\begin{vmatrix} g, & 0 \\ -l, & \frac{n}{g}\end{vmatrix}$, the first equation becomes

$$\frac{g}{nM}Q(\Omega) = Q(\omega) - 2k^2(1-k^2)\frac{K(\omega)}{M}\frac{dM}{d.k^2},$$

which, on eliminating g by the equation $g = \dfrac{K(\omega)}{MK(\Omega)}$ and substituting for $2k^2k'^2\dfrac{d}{d.k^2}$ its equivalent $\dfrac{\lambda\lambda'^2}{nM^2}\dfrac{d}{d\lambda}$, coincides with equation (ii) i.e. with equation (i). Thus the equation of Prof. Cayley holds for all *reduced* transformations, i.e. for all transformations of which the second element of the matrix of transformation is $= 0$. This remark is of some importance as it enables us to show that the equation

$$\frac{d^2\Sigma}{dx^2} + 2n\frac{Q(\omega)}{K(\omega)}x\frac{d\Sigma}{dx} + 4nk^2(1-k^2)\frac{d\Sigma}{dk} = 0,$$

which is satisfied by $\Sigma = \vartheta_0\left[\dfrac{\pi x}{2MK(x)}, \Omega\right]$ when $\omega = \dfrac{\Omega}{n}$, i.e. when the transformation is the first transformation of Jacobi, is also satisfied by the same function when the transformation considered is any reduced transformation whatever.

By combining the foregoing results we can express the elements of the matrix of transformation in terms of the multiplier and of the integrals K, K', Q, Q'; we thus find

$$\text{(iii)}\quad -\tfrac{1}{2}i\pi\begin{vmatrix}\alpha, & \beta \\ \gamma, & \delta\end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{M}K(\omega), & nMQ(\omega) - 2nk^2(1-k^2)\frac{dM}{d.k^2}K(\omega) \\ \frac{1}{M}iK'(\omega), & nMiQ'(\omega) - 2nk^2(1-k^2)\frac{dM}{d.k^2}iK'(\omega)\end{vmatrix}$$

$$\times\begin{vmatrix} iQ'(\Omega), & -Q(\Omega) \\ -iK'(\Omega), & K(\Omega)\end{vmatrix},$$

and the equation of Prof. Cayley is then found by putting $\beta=0$ in the formula.

If $\gamma=0$, we have

$$\text{(iv)} \qquad \frac{Q'(\omega)}{K'(\omega)}=\frac{1}{nM'}\left[\frac{Q'(\Omega)}{K'(\Omega)}+\frac{\lambda\lambda'^2}{M}\frac{dM}{d\lambda}\right].$$

When β and γ are each $=0$, i.e. in the real transformation $\omega=\begin{vmatrix} \alpha, & 0 \\ 0, & \delta \end{vmatrix}\times\Omega$, a combination of the foregoing equations gives $K(\omega)K'(\omega)=nM^2K(\Omega)K'(\Omega)$, which is true because

$$\frac{K(\omega)}{M}=\alpha K(\Omega), \quad \frac{iK'(\omega)}{M}=i\delta K'(\Omega).$$

IV.

Employment of the Transformation of the Function of the second species to obtain expressions rational in k^2 and λ^2 for the coefficients in the formula of Transformation of the n^{th} order.

The analytical theory of transformation supplies a direct proof that the coefficients a, b in the developments

$$1+a_1x^2+a_2x^4\ldots+a_{\frac{1}{2}(n-1)}x^{n-1}=\Pi_j\left[1-\frac{x^2}{\sin^2\text{am}\,4j\eta}\right]$$

and $$1+b_1x^2+b_2x^4\ldots+b_{\frac{1}{2}(n-1)}x^{n-1}=\Pi_j\left[1-k^2x^2\sin^2\text{am}\,4j\eta\right]$$

of the numerator and denominator functions in the formula of transformation of the n^{th} order are rational in k^2, λ^2, M, and $\frac{dM}{dk^2}$; that is in k^2, λ^2 and M, for $\frac{dM}{dk^2}$ is always rational in k^2, λ^2 and M: this follows from the equation of Jacobi combined with the theorem that at a multiple point on the modular curve the branches are all linear, and the tangents all different; so that even in the case where the modular equation has equal roots the coefficients contain no irrationalities other than those involved in the expression of M.

To obtain the proof it is convenient to consider the function of the second species defined by the equation

$$Zu=\int_0^u k^2\sin^2\text{am}\,u\,du,$$

and expressed by Jacobi in the form

$$Zu = \frac{J}{K} u - \frac{d}{du} \log \vartheta_0 \left(\frac{\pi u}{2K}\right),$$

where $J = Z(K) = \int_0^K k^2 \sin^2 \mathrm{am}\, u\, du$. Employing the usual notation $\omega = \frac{iK'}{K}$, $q = e^{i\pi\omega}$, and regarding $\sqrt[4]{k} = \psi(\omega)$, $K = K(\omega)$, $J = J(\omega)$ as one-valued functions of ω defined by the equations

$$(1) \quad \begin{cases} \sqrt[4]{k} = \sqrt{2} q^{\frac{1}{8}} \prod \dfrac{1+q^{2m}}{1+q^{2m-1}}, \\ \sqrt[4]{k'} = \prod \dfrac{1-q^{2m-1}}{1+q^{2m-1}}, \\ \sqrt{\left(\dfrac{2K}{\pi}\right)} = 1 + 2q + 2q^4 + 2q^9 + \ldots, \\ J = k^2 K - \frac{1}{2} i\pi \dfrac{d}{d\omega} \dfrac{1}{K}; \end{cases}$$

we consider the transformation

$$\omega = \begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix} \times \Omega, = \frac{\gamma + \delta\Omega}{\alpha + \beta\Omega},$$

where $\begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix} \equiv \begin{vmatrix} 1, & 0 \\ 0, & 1 \end{vmatrix}$ (mod 2), and $\alpha\delta - \beta\gamma = n$.

Defining the multiplier M by the equations

$$(2) \quad \begin{cases} \dfrac{1}{M} K(\omega) = \alpha K(\Omega) + i\beta K'(\Omega), \\ \dfrac{1}{M} iK'(\omega) = \gamma K(\Omega) + i\delta K'(\Omega), \end{cases}$$

we write in the equation of Jacobi $\frac{u}{M}$ for u and Ω for ω, Dividing by M we obtain

$$(3) \quad \frac{1}{M} Z\left(\frac{u}{M}, \lambda^2\right) = \frac{J(\Omega)}{K(\Omega)} \frac{u}{M^2} - \frac{d}{du} \log \vartheta_0 \left[\frac{\pi u}{2MK(\Omega)}, \Omega\right]$$

$$\left(= \frac{J(\Omega)}{K(\Omega)} \frac{u}{M^2} - \frac{d}{du} \log \vartheta_0 \left[\frac{\pi u (\alpha + \beta\Omega)}{2K(\omega)}, \Omega\right]\right),$$

where $\lambda^2 = \phi^8(\Omega)$. With this equation we combine a known formula for the transformation of the Theta functions, viz.

$$(4) \quad \vartheta_0\left[\frac{(\alpha+\beta\Omega)\pi u}{2K(\omega)}, \Omega\right] = C \times \vartheta_0^n\left[\frac{\pi u}{2K(\omega)}, \omega\right]$$

$$\times e^{-i\pi\beta(\alpha+\beta\Omega)\frac{u^2}{4K^2}} \times \prod_{j=1}^{j=\frac{1}{2}(n-1)} [1 - k^2 \sin^2\text{am} 4j\eta \sin^2\text{am} u],$$

in which $C = \vartheta_0[\Omega] \div \vartheta_0^n[\omega]$, and $\eta = \dfrac{\mu K + i\nu K'}{n}$, ν being the greatest common measure of α and β, and μ being defined by the congruence $\mu \equiv \gamma r + \delta s \left(\text{mod } \dfrac{n}{\nu}\right)$, in which r and s are any two integers satisfying the equation $\alpha r + \beta s = \nu$.

Instead of η we may take any multiple of η by a number prime to n. Taking the logarithm and differentiating, we obtain from (4) the equation

$$(5) \quad \frac{d}{du}\log\vartheta_0\left[\frac{(\alpha+\beta\Omega)\pi u}{2K}, \Omega\right]$$

$$= \frac{2\pi\beta(\alpha+\beta\Omega)u}{2K^2} + n\frac{d}{du}\log\vartheta_0\left[\frac{\pi u}{2K}, \omega\right]$$

$$+ \sum_{j=1}^{j=\frac{1}{2}(n-1)} \frac{d}{du}\log[1 - k^2\sin^2\text{am}\,\eta \sin^2\text{am}\,u],$$

or finally the equation

$$\frac{1}{M}Z\left(\frac{u}{M}, \lambda^2\right) - n\,Z(u, k^2) + 2Hu$$

$$= 2k^2 \sin\text{am}\,u \cos\text{am}\,u\,\Delta\,\text{am}\,u$$

$$\times \sum_{j=1}^{j=\frac{1}{2}(n-1)} \frac{\sin^2\text{am}\,4j\eta}{1 - k^2\sin^2\text{am}\,4j\eta\sin^2\text{am}\,u};$$

where

$$(6) \qquad 2H = \frac{nJ(\omega)}{K(\omega)} - \frac{1}{M^2}\frac{K(\Omega)}{J(\Omega)} - \frac{2\pi\beta(\alpha+\beta\Omega)}{2K^2(\omega)}$$

$$= \frac{nJ(\omega)}{K(\omega)} - \frac{1}{M^2}\frac{J(\Omega)}{K(\Omega)} - \frac{2\pi\beta}{2MK(\omega)K(\Omega)}.$$

The formulæ (5) and (6) are given in fact by Jacobi for the transformation of the elliptic function of the second species.

For our present purpose we have to show however that the constant H is a rational function of k^2, λ^2, M, $\frac{dM}{d.k^2}$; and that the integrals $J(\omega)$, $J(\Omega)$, $K(\omega)$, $K(\Omega)$ enter into the expression (6) only apparently.

To verify the assertion we write $-Q(\omega)+k^2K(\omega)$ for $J(\omega)$, and consequently $-Q(\Omega)+\lambda^2K(\Omega)$ for $J(\Omega)$. Similarly for the function $J(\omega)$ defined by the equation

$$(7) \qquad K(\omega)J'(\omega)-K'(\omega)J(\omega)=\tfrac{1}{2}\pi,$$

we write $\qquad -Q'(\omega)+k^2K'(\omega)$

so that

$$(8) \qquad K'(\omega)\,Q(\omega)-K(\omega)\,Q'(\omega)=\tfrac{1}{2}\pi.$$

Then we have

$$(9) \qquad \frac{1}{nM}[\alpha Q(\Omega)+i\beta Q'(\Omega)]=Q(\omega)-2k^2(1-k^2)\frac{K(\omega)}{M}\frac{dM}{d.k^2},$$

$$\frac{1}{nM}[\gamma Q(\Omega)+i\delta Q'(\Omega)]=iQ'(\omega)-2k^2(1-k^2)\frac{K'(\omega)}{M}\frac{dM}{d.k^2},$$

which are the general formulæ for the transformation of the complete integrals of the second kind represented by

$$Q(\omega)=\int_0^K k^2\cos^2\mathrm{am}\,x\,dx,$$

$$iQ'(\omega)=\int_0^{K+iK'} k^2\cos^2\mathrm{am}\,x\,dx.$$

Introducing the functions $Q(\omega)$, $Q'(\omega)$ into the equation (6), we have

$$(10) \qquad 2H=nk^2-\frac{\lambda^2}{M^2}-\frac{nQ(\omega)}{K(\omega)}+\frac{1}{M^2}\frac{Q(\Omega)}{K(\Omega)}-\frac{i\pi b}{2MK(\omega)K(\Omega)}.$$

But from (9)

$$\frac{Q(\Omega)}{M^2K(\Omega)}=\frac{\delta Q(\omega)-i\beta Q'(\omega)}{MK(\Omega)}-\frac{2nk^2(1-k^2)}{M^2}\frac{dM}{d.k^2}.$$

Substituting this value in (10) we obtain

$$2H=nk^2-\frac{\lambda^2}{M^2}-\frac{2nk^2(1-k^2)}{M}\frac{dM}{d.k^2}$$

$$-\frac{1}{MK(\omega)K(\Omega)}[nMK(\Omega)Q(\omega)-K(\omega)\{\delta Q(\omega)-i\beta Q'(\omega)\}-\tfrac{1}{2}i\pi\beta],$$

or finally

$$(11)\qquad 2H = nk^2 - \frac{\lambda^2}{M^2} - \frac{2nk^2(1-k^2)}{M}\frac{dM}{d.k^2},$$

the terms in the square brackets vanishing by virtue of the equations

$$nMK(\Omega) = \delta K(\omega) - i\beta K'(\omega);\ K'(\omega)\,Q(\omega) - K(\omega)\,Q'(\omega) = \tfrac{1}{2}\pi.$$

If we now employ the developments

$$\sin^2\text{am}\,u = u^2 - \tfrac{1}{3}(1+k^2)\,u^4 + \tfrac{1}{45}(2+13k^2+2k^4)\,u^6 + \ldots$$

$$\frac{d.\sin^2\text{am}\,u}{du} = 2u - \tfrac{4}{3}(1+k^2)\,u^3 + \tfrac{2}{15}(2+13k^2+2k^4)\,u^5 + \ldots$$

$$Zu = \tfrac{1}{3}k^2u^3 - \tfrac{1}{15}k^2(1+k^2)\,u^5 + \tfrac{1}{315}k^2(2+13k^2+3k^4)\,u^7 + \ldots$$

(of which, however, the general terms are not known, and which are convergent only for values of u not surpassing a given certain limit) we can by equating the coefficients of like powers of u in the two members of equation (5) obtain successively the values of the sums

$$\sum_{j=1}^{j=\frac{1}{2}(n-1)} k^{2s}\sin^{2s}\text{am}\,4j\eta,$$

expressed rationally in terms of λ^2, k^2 and H. Thus for example

$$\Sigma k^2\sin^2\text{am}\,4j\eta = H,$$

$$\Sigma k^4\sin^4\text{am}\,4j\eta = \frac{\lambda^2}{6M^4} - \frac{\Delta k^2}{6} + \tfrac{1}{3}(1+k^2)\,H$$

$$= \tfrac{1}{6}\left[\frac{\lambda^2}{M^4} - 2\lambda^2(1+k^2)\frac{1}{M^2} + \Delta k^2(1+2k^2)\right.$$

$$\left. - 4nk^2(1-k^2)(1+k^2)\frac{1}{M}\frac{dM}{d.k^2}\right].$$

But from the theory of transformation

$$1 + b_1x^2 + b_2x^4 \ldots + b_{\frac{1}{2}(n-1)}x^{n-1} = \Pi_j(1 - k^2x^2\sin^2\text{am}\,4j\eta),$$

so that the coefficients b_1, $b_2\ldots$ are rational because the

quantities $\Sigma k^{2s} \sin^{2s}\mathrm{am}\, 4j\eta$ are so. In particular

$$b_1 = \frac{nk^2(1-k^2)}{M}\frac{dM}{d.k^2} + \tfrac{1}{2}\frac{\lambda^2}{M^2} - \tfrac{1}{2}nk^2,$$

$$b_2 = \tfrac{1}{24}\frac{\lambda^2(3\lambda^2-2)}{M^4} - \tfrac{1}{12}\frac{\lambda^2\{(3n-2)k^2-2\}}{M^2} + \frac{n\lambda^2k^2(1-k^2)}{2M^3}\frac{dM}{d.k^2}$$
$$- \frac{nk^2(1-k^2)\{(3n-2)k^2-2\}}{6M}\frac{dM}{d.k^2}$$
$$+ \tfrac{1}{2}\frac{n^2k^4(1-k^2)^2}{M^2}\left(\frac{dM}{d.k^2}\right)^2 + \frac{nk^2\{(3n-4)k^2-2\}}{24}.$$

The last coefficient is known, viz. we have

$$b_{\frac{1}{2}(n-1)} = (-)^{\frac{1}{2}(n-1)}k^{n-1}\prod_{j=1}^{j=\frac{1}{2}(n-1)} \sin^2\mathrm{am}\, 4j\eta = \frac{k^{\frac{1}{2}(n-1)}\sqrt{\lambda}}{M},$$

the expression being rational in k^2 and λ^2.

Lastly the coefficients a are rational because

$$1 + a_1x^2 + a_2x^4 \ldots + a_{\frac{1}{2}(n-1)}x^{n-1} = \prod_{j=1}^{j=\frac{1}{2}(n-1)}\left[1 - \frac{x^2}{\sin^2\mathrm{am}\, 4j\eta}\right],$$

whence it follows that

$$\frac{a_s}{k^{2s}} = \frac{M}{k^{\frac{1}{2}(n-1)}\sqrt{\lambda}}\, b_{\frac{1}{2}(n-1)-s}.$$

V.

Relations in the formulæ of Jacobi between the elements η and the transformation represented by the formula $\omega = \dfrac{\gamma+\delta\Omega}{\alpha+\beta\Omega}$.

The general formulæ of Jacobi for any transformation of an uneven order n are

$$\text{(i)}\left\{\begin{aligned}
&\sin\mathrm{am}\left(\frac{u}{M}, \lambda^2\right) = \frac{\sin\mathrm{am}\, u}{MD}\prod_{j=1}^{j=\frac{1}{2}(n-1)}\left[1 - \frac{\sin^2\mathrm{am}\, u}{\sin^2\mathrm{am}\, 4j\eta}\right],\\
&\cos\mathrm{am}\left(\frac{u}{M}, \lambda^2\right) = \frac{\cos\mathrm{am}\, u}{D}\prod_{j=1}^{j=\frac{1}{2}(n-1)}\left[1 - \frac{\sin^2\mathrm{am}\, u}{\sin^2\mathrm{coam}\, 4j\eta}\right],\\
&\Delta\,\mathrm{am}\left(\frac{u}{M}, \lambda^2\right)\\
&\qquad = \frac{\Delta\,\mathrm{am}\, u}{D}\prod_{j=1}^{j=\frac{1}{2}(n-1)}\left[1 - k^2\sin^2\mathrm{am}\, u\,\sin^2\mathrm{coam}\, 4j\eta\right],\\
&\qquad D = \prod_{j=1}^{j=\frac{1}{2}(n-1)}\left[1 - k^2\sin^2\mathrm{am}\, u\,\sin^2\mathrm{am}\, 4j\eta\right];
\end{aligned}\right.$$

$$
\text{(ii)}\quad\left\{\begin{aligned}
\lambda^2 &= k^{2n} \times \prod_{j=1}^{j=\frac{1}{2}(n-1)} \left[\frac{\cos^2 \operatorname{am} 4j\eta}{\Delta^2 \operatorname{am} 4j\eta}\right]^4,\\
\lambda'^2 &= k'^{2n} \times \prod_{j=1}^{j=\frac{1}{2}(n-1)} \left[\frac{1}{\Delta^2 \operatorname{am} 4j\eta}\right]^4,\\
\frac{1}{M^2} &= \frac{k^n}{\lambda} \times \prod_{j=1}^{j=\frac{1}{2}(n-1)} \sin^4 \operatorname{am} 4j\eta\\
&\qquad\qquad = \prod_{j=1}^{j=\frac{1}{2}(n-1)} \frac{\sin^4 \operatorname{am} 4j\eta}{\sin^4 \operatorname{coam} 4j\eta};
\end{aligned}\right.
$$

η representing an element of the form

$$
\text{(iii)}\qquad \frac{pK+iqK'}{n},
$$

where p and q are integral numbers having no common divisor with n.

It is also known that the modules k^2 and λ^2 are connected by a relation of the form

$$
\text{(iv)}\qquad \left\{\begin{aligned}
&k^2 = \phi^8(\omega), \quad \lambda^2 = \phi^8(\Omega),\\
&\omega = \frac{\gamma+\delta\Omega}{\alpha+\beta\Omega} = \begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix} \times \Omega,
\end{aligned}\right.
$$

where $\begin{Vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{Vmatrix}$ is a primary and primitive matrix of integral numbers, having n for its determinant; and $\phi(\omega)$ is the function already defined in Note I.

The elliptic transformation is completely determined when either (1) the integral numbers (p, q), or (2) the integral numbers $\alpha, \beta, \gamma, \delta$ are given. Thus the two converse problems present themselves (1) "given any pair (p, q), to find all the corresponding matrices $\begin{Vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{Vmatrix}$"; and (2) "given any matrix $\begin{Vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{Vmatrix}$, to find all the corresponding pairs (p, q)."

The solution of these problems is implicitly contained in the congruences

$$
\text{(v)}\qquad p\alpha+q\gamma \equiv 0, \quad p\beta+q\delta \equiv 0, \quad \text{mod } n,
$$

which result from the theory of the transformation of the Theta functions; but for the sake of clearness it may be worth while to give the details of the solution, although the arithmetical considerations involved are quite elementary.

(*a*) If $p_1 \equiv p_2$, $q_1 \equiv q_2$, mod n, the pairs (p_1, q_1), (p_2, q_2), and the elements η_1, η_2, may be said to be *congruous*; if $hp_1 \equiv p_2$, $hq_1 \equiv q_2$, mod. n, where h is an integral number prime to n, the pairs (p_1, q_1), (p_2, q_2), and the elements η_1, η_2, may be said to be *equivalent*. Congruous elements are always equivalent; and equivalent elements, when employed in the formulæ of Jacobi, give identical results.

(*b*) To find, for any given uneven number n, the number N of non-equivalent elements η, let $\chi(s)$ denote the number of numbers prime to any given number s, and not surpassing s, and let δ, δ' be two relatively prime divisors of n. The number of incongruous elements η, having δ and δ' for the greatest common divisors of n with p, and of n with q respectively, is $\chi\left(\frac{n}{\delta}\right) \times \chi\left(\frac{n}{\delta'}\right)$; hence the number of non-equivalent elements, having their greatest common divisors, is

$$\text{(vi)} \qquad \chi\left(\frac{n}{\delta}\right) \times \chi\left(\frac{n}{\delta'}\right) \div \chi(n) = \chi\left(\frac{n}{\delta\delta'}\right);$$

and the whole number N of non-equivalent elements is given by the formula

$$N = \Sigma\chi\left(\frac{n}{\delta\delta'}\right),$$

the summation extending to every pair of relatively prime divisors of n, and the pairs δ, δ' and δ', δ being regarded as different, except when $\delta' = \delta = 1$. It is easy to simplify the expression thus obtained for N; viz. if $n = A \times B$, A and B being two numbers relatively prime, we find

$$\Sigma\chi\left(\frac{n}{\delta\delta'}\right) = \Sigma\chi\left(\frac{A}{\delta\delta'}\right) \times \Sigma\chi\left(\frac{B}{\delta\delta'}\right);$$

also, if $n = \theta^\mu$, θ being a prime,

$$\Sigma\chi\left(\frac{n}{\delta\delta'}\right) = \theta^\mu\left(1 + \frac{1}{\theta}\right).$$

Hence, in general

$$N = \Sigma\chi\left(\frac{n}{\delta\delta'}\right) = n\Pi\left(1 + \frac{1}{\theta}\right);$$

the sign of multiplication extending to all the primes θ dividing n.

(c) Two matrices $\begin{Vmatrix} \alpha_1, & \beta_1 \\ \gamma_1, & \delta_1 \end{Vmatrix}$ and $\begin{Vmatrix} \alpha_2, & \beta_2 \\ \gamma_2, & \delta_2 \end{Vmatrix}$ are said to be equivalent (or, more properly, equivalent by primary post-multiplication), when

$$\begin{Vmatrix} \alpha_1, & \beta_1 \\ \gamma_1, & \delta_1 \end{Vmatrix} = \begin{Vmatrix} \alpha_2, & \beta_2 \\ \gamma_2, & \delta_2 \end{Vmatrix} \times \| \varepsilon \|;$$

$\| \varepsilon \|$ being a primary unit matrix.

Equivalent matrices, employed in the formulæ (iv), give identical values for λ^2.

(d) The number of primitive and primary matrices $\begin{Vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{Vmatrix}$ non-equivalent by primary post-multiplication, is also $n\Pi\left(1+\frac{1}{\theta}\right)$; so that there are as many non-equivalent matrices as there are non-equivalent elements η; this we know, *à priori*, must be the case; because the number of different transformations must be equal to the number of non-equivalent matrices, and also to the number of non-equivalent elements.

(e) We may take as representatives of the N non-equivalent elements η the elements of the *reduced* system

$$\zeta = \frac{-2lK + igK'}{n},$$

where g is any divisor of n, and $2l$ is any term of a system of residues, even and prime to g, taken with respect to the modulus $\frac{n}{g}$. No two of the elements ζ can be equivalent; for if $2hl_1 \equiv 2hl_2$, mod n; $hg_1 \equiv hg_2$, mod n; we must have in the first place $g_1 = g_2$, because h is prime to n; and in the second place, writing g for g_1 or g_2, $h \equiv 1$, mod $\frac{n}{g}$; $l_1 \equiv l_2$, mod $\frac{n}{g}$; i.e. $l_1 = l_2$, and the two elements ζ, which were supposed equivalent, are identical. Again, any given element $\frac{pK+iqK'}{n}$ is equivalent to one of the elements ζ; for, if g be the greatest common divisor of q and n, let $-2l$ be the residue of $\frac{n}{g}$, which satisfies the congruence $-2l\frac{q}{g} \equiv p$, mod $\frac{n}{g}$, and which may be supposed prime to g,

because p is prime to g; then the simultaneous congruences

$$\left.\begin{aligned} hp &\equiv -2l \\ hq &\equiv \quad g \end{aligned}\right\}, \text{ mod } n,$$

are resoluble, because p, q, and n are relatively prime, and because the determinant $pg+2lq$ is divisible by n; further, the value of h supplied by these congruences is prime to n, because $2l$ and g are relatively prime; thus the given element is equivalent to the reduced element $\dfrac{-2lK+igK'}{n}$.

Example.

Let $\eta=\dfrac{581K+4000iK'}{4725}$ be the given element. Here $n=4725$, $g=25$, $\dfrac{n}{g}=189$, $\dfrac{q}{g}=160$, $-2l\times 160\equiv 581$, mod 189; whence $-2l\equiv 182$, mod 189. The congruences $581h\equiv 182$, $4000h\equiv 25$, mod 4725, give $h\equiv 1147$, mod 4725; so that the reduced element $\dfrac{182K+25iK'}{4725}$ is congruous to $1147\times\eta$, and consequently equivalent to η.

(f) As representatives of the non-equivalent matrices of determinant n, we may take the matrices

$$\| \zeta \| = \begin{Vmatrix} g, & 0 \\ 2l, & \dfrac{n}{g} \end{Vmatrix},$$

g and $2l$ having the same significations as in (e).

(g) The reduced pair $(-2l, g)$, and the reduced matrix $\begin{Vmatrix} g, & 0 \\ 2l, & \dfrac{n}{g} \end{Vmatrix}$, satisfy the congruences (v); and the reduced pairs and matrices cannot be combined in any other way so as to satisfy those congruences.

(h) The congruences (v), which may also be written in the symbolic form

$$(p, q)\times \begin{Vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{Vmatrix} \equiv 0, \text{ mod } n,$$

admit, for any given primitive matrix $\begin{Vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{Vmatrix}$, $\psi(n)$ primitive and incongruous solutions (p, q); which, however, are all equivalent: and if we replace the matrix by any equivalent matrix, these solutions remain unchanged.

From these considerations it is evident that, to solve the two converse problems, we have only to determine the reduced pair equivalent to the given pair, or the reduced matrix equivalent to the given matrix, as the case may be; either of them being known, the other is known also; and the proposed problem is completely solved.

Example.

If $n = 4725$, and $(581, 4000)$ is the given pair, the equivalent reduced pair is (as we have already seen) $(182, 25)$; the corresponding reduced matrix is therefore $\begin{Vmatrix} 25, & 0 \\ -182, & 189 \end{Vmatrix}$, and all the corresponding matrices are included in the formula

$$\begin{Vmatrix} 25, & 0 \\ -182, & 189 \end{Vmatrix} \times \| \varepsilon \|,$$

where $\| \varepsilon \|$ is a primary unit matrix.

Again, if $\begin{Vmatrix} 75, & 100 \\ -168, & -161 \end{Vmatrix}$ is a given matrix of the determinant 4725, the equivalent reduced matrix is $\begin{Vmatrix} 25, & 0 \\ -182, & 189 \end{Vmatrix}$; hence $(182, 25)$ is the corresponding reduced pair, and all the corresponding pairs are included in the formula

$$(182h + 4725a, \quad 25h + 4725b),$$

where a, b, h are any integers, positive and negative, of which h is prime to 4725.

If we introduce the one-valued functions

$$\sqrt{k} = \phi^2(\omega), \quad \sqrt{\lambda} = \phi^2(\Omega), \quad \sqrt{k'} = \psi^2(\omega), \quad \sqrt{\lambda'} = \psi^2(\Omega),$$

we may write the first two of the equations (ii) in the form

$$\text{(vii)} \qquad \left\{ \begin{aligned} \phi^2(\Omega) &= i^{-\frac{1}{2}\gamma\delta} \phi^{2n}(\omega) \prod \frac{\cos^2 \operatorname{am} 4j\eta}{\Delta^2 \operatorname{am} 4j\eta}, \\ \psi^2(\Omega) &= i^{\frac{1}{2}\alpha\beta}\, \psi^{2n}(\omega) \prod \frac{1}{\Delta^2 \operatorname{am} 4j\eta}. \end{aligned} \right.$$

Similarly the last of the equations (ii) may be written

$$
\text{(viii)}\quad\left\{
\begin{aligned}
\frac{1}{M} &= (-1)^{\frac12(\delta-1)i-\frac12\gamma\delta}\,\frac{\phi^{2n}(\omega)}{\phi^2(\Omega)}\,\Pi \sin^2\mathrm{am}\,4j\eta, \\
&= (-1)^{\frac12(\delta-1)} \prod \frac{\sin^2\mathrm{am}\,4j\eta}{\sin^2\mathrm{coam}\,4j\eta}.
\end{aligned}
\right.
$$

In the theory of the transformation of the Theta functions it is convenient to suppose that the matrix $\begin{Vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{Vmatrix}$ satisfies the congruences $\beta\equiv 0$, $\gamma\equiv 0$, mod. 8, $\alpha\equiv 1$, mod. 4. In this case the equations (vii) and (viii) present themselves in the form

$$\sqrt{\lambda} = k^{\frac12 n} \prod \frac{\cos^2\mathrm{am}\,4j\eta}{\Delta^2\mathrm{am}\,4j\eta},$$

$$\sqrt{\lambda'} = k'^{\frac12 n} \prod \frac{1}{\Delta^2\mathrm{am}\,4j\eta},$$

$$\frac{1}{M} = (-1)^{\frac12(n-1)}\,\frac{k^{\frac12 n}}{\sqrt{\lambda}}\,\Pi \sin^2\mathrm{am}\,4j\eta = (-1)^{\frac12(n-1)} \prod \frac{\sin^2\mathrm{am}\,4j\eta}{\sin^2\mathrm{coam}\,4j\eta},$$

in which they occur in the *Fundamenta Nova*. From these special formulæ, in which $\sqrt{\lambda}$, $\sqrt{\lambda'}$, $\sqrt{k}$, $\sqrt{k'}$ represent the one-valued functions $\phi^2(\Omega)$, $\psi^2(\Omega)$, $\phi^2(\omega)$, $\psi^2(\omega)$, the general formula (vii) and (viii) may be immediately inferred by linear transformation.

If $\frac{n}{g}=\gamma\gamma'$, where γ is the greatest divisor of $\frac{n}{g}$ which is prime to g, the number of the residues of $\frac{n}{g}$ which are prime to g (*i.e.* the number of different numbers $2l$ in (*e*) *suprà* p. 14) is $\gamma\chi(\gamma')$. Hence, the whole number of non-equivalent pairs is $\Sigma\gamma\chi(\gamma')$, the summation extending to every divisor g of n. We have, consequently,

$$\Sigma\gamma\chi(\gamma') = \Sigma\chi\left(\frac{n}{\delta\delta'}\right),$$

$$\Sigma\gamma\chi(\gamma') = n\,\Pi\left(1+\frac{1}{\theta}\right);$$

these results are easily verified independently (see "A Memoir on the Singularities of the Modular Curves," *Proceedings of the London Mathematical Society*, vol. IX. [1878], Art. 9).

It may deserve notice that the equation (vi) is the first of a series of elementary formulæ of the same general type. The next in order is

$$\chi(n) \times \chi\left(\frac{n}{\delta_2\delta_3}\right)\chi\left(\frac{n}{\delta_3\delta_1}\right)\chi\left(\frac{n}{\delta_1\delta_2}\right)$$
$$= \chi\left(\frac{n}{\delta_1}\right)\chi\left(\frac{n}{\delta_2}\right)\chi\left(\frac{n}{\delta_3}\right) \times \chi\left(\frac{n}{\delta_1\delta_2\delta_3}\right);$$

the general formula being

$$\chi(n) \times \prod \chi\left(\frac{n}{\delta_1\delta_2}\right) \times \prod \chi\left(\frac{n}{\delta_1\delta_2\delta_3\delta_4}\right) \times \ldots$$
$$= \prod \chi\left(\frac{n}{\delta}\right) \times \prod \chi\left(\frac{n}{\delta_1\delta_2\delta_3}\right) \times \ldots$$

The s divisors $\delta_1, \delta_2, \ldots \delta_s$ are not necessarily relatively prime; but, if p be any prime dividing σ of them, and having a for its exponent in n, and b for the sum of its exponents in $\delta_1, \delta_2, \ldots$, we must have either $b < a$, or else $b = a$, $\sigma < s$. If $b = a$, $\sigma = s$, the factor $\frac{\chi(p)}{p} = \frac{p-1}{p}$ must be applied to the left or right-hand member according as s is even or uneven.

VI.

Further Theory of the Functions $Q(\omega)$, $Q'(\omega)$.

[In explanation of the notation, observe that every matrix of uneven determinant is, with regard to the modulus 2, of one of the six types

$$\begin{vmatrix} 1, & 0 \\ 0, & 1 \end{vmatrix}, \quad \begin{vmatrix} 0, & 1 \\ -1, & 0 \end{vmatrix}, \quad \begin{vmatrix} 1, & 0 \\ 1, & 1 \end{vmatrix}, \quad \begin{vmatrix} 1, & 1 \\ 0, & 1 \end{vmatrix}, \quad \begin{vmatrix} 1, & 1 \\ -1, & 0 \end{vmatrix}, \quad \begin{vmatrix} 0, & 1 \\ -1, & -1 \end{vmatrix},$$

which are represented by the symbols

$$1, \ \psi, \ \sigma, \ \tau, \ \rho, \ \rho^2;$$

and considering these as unit matrices, i.e. supposing $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$ to be equal instead of only congruent to the foregoing values, they are represented by

$$|1|, \ |\psi|, \ |\sigma|, \ |\tau|, \ |\rho|, \ |\rho^2|.$$

Similarly the primitive matrices of any even determinant, considered with regard to the modulus 2, are of one or other

of the following nine types:

$$\begin{vmatrix}1, & 1\\ 1, & 1\end{vmatrix}, \quad \begin{vmatrix}1, & 0\\ 1, & 0\end{vmatrix}; \quad \begin{vmatrix}0, & 1\\ 0, & 1\end{vmatrix},$$

$$\begin{vmatrix}0, & 0\\ 1, & 1\end{vmatrix}, \quad \begin{vmatrix}0, & 0\\ 1, & 0\end{vmatrix}, \quad \begin{vmatrix}0, & 0\\ 0, & 1\end{vmatrix};$$

$$\begin{vmatrix}1, & 1\\ 0, & 0\end{vmatrix}; \quad \begin{vmatrix}1, & 0\\ 0, & 0\end{vmatrix}, \quad \begin{vmatrix}0, & 1\\ 0, & 0\end{vmatrix};$$

which are symbolized thus

$$\begin{vmatrix} C_{1,1}, & C_{2,1}, & C_{3,1} \\ C_{1,2}, & C_{2,2}, & C_{3,2} \\ C_{1,3}, & C_{2,3}, & C_{3,3} \end{vmatrix}.$$

The functions $Q(\omega)$ and $Q'(\omega)$ have opposite signs to the $Q(\omega)$ and $Q'(\omega)$ defined in Note III. (p. 1). The quantity η denotes $e^{\frac{1}{8}i\pi}$.]

The following is a Table of the linear transformations of $Q(\omega)$ and $Q'(\omega)$. For brevity the argument ω is omitted, and the functions $J = Q + k^2K$, $J' = Q' + k^2K'$ are introduced:

$$\omega = \frac{c + d\Omega}{a + b\Omega}.$$

	$\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$	$\frac{1}{M}[aQ(\Omega) + ibQ'(\Omega)]$	$\frac{1}{M}[cQ(\Omega) + idQ'(\Omega)]$	$\frac{1}{M}$
	$\equiv$	$=$	$=$	$=$
I	$\lvert 1 \rvert$	Q	iQ'	$(-1)^{\frac{a-1}{2}}$
II	$\lvert \psi \rvert$	Q	iQ'	$(-1)^{\frac{b-1}{2}}i$
III	$\lvert \sigma \rvert$	J	iJ'	$(-1)^{\frac{a-1}{2}}k'$
IV	$\lvert \tau \rvert$	$J - K$	$i(J' - K')$	$(-1)^{\frac{a+c-1}{2}}k$
V	$\lvert \rho \rvert$	$J - K$	$i(J' - K')$	$(-1)^{\frac{c+1}{2}}ik$
VI	$\lvert \rho^2 \rvert$	J	iJ'	$(-1)^{\frac{b-1}{2}}ik'$

We add a list of useful particular cases; the formulæ are immediately deducible from the Table. The functions $P(\omega)$ and $P'(\omega)$ represent $J(\omega) - k'^2K(\omega)$ and $J'(\omega) - k'^2K'(\omega)$ respectively. For convenience, the transformations of K and K' are included in the list, and as before the argument ω is omitted.

$$\text{II.}\quad \omega = |\psi| \times \Omega = -\frac{1}{\Omega}.$$

$$\begin{aligned}
K(\Omega) &= K'; & iK'(\Omega) &= iK;\\
J(\Omega) &= -J'+K'; & iJ'(\Omega) &= -iJ+iK;\\
P(\Omega) &= -P'; & iP'(\Omega) &= -iP;\\
Q(\Omega) &= -Q'; & iQ'(\Omega) &= -iQ.
\end{aligned}$$

$$\text{III.}\quad \omega = \begin{vmatrix} 1, & 0 \\ \mp 1, & 1 \end{vmatrix} \times \Omega = \mp 1 + \Omega.$$

$$\begin{aligned}
K(\Omega) &= k'K; & iK'(\Omega) &= k'(\pm K + iK');\\
J(\Omega) &= \frac{1}{k'}Q; & iJ'(\Omega) &= \frac{1}{k'}(\pm Q + iQ');\\
P(\Omega) &= \frac{1}{k'}(Q-K); & iP'(\Omega) &= \frac{1}{k'}[\pm(Q-K) + i(Q'-K')];\\
Q(\Omega) &= \frac{J}{k'}; & iQ'(\Omega) &= \frac{1}{k'}(\pm J + iJ').
\end{aligned}$$

$$\text{IV.}\quad \omega = \begin{vmatrix} 1, & \mp 1 \\ 0, & 1 \end{vmatrix} \times \Omega = \frac{\Omega}{1 \mp \Omega}.$$

$$\begin{aligned}
K(\Omega) &= k(K \pm iK'); & iK'(\Omega) &= ikK';\\
J(\Omega) &= \frac{1}{k}(J \pm iJ'); & iJ'(\Omega) &= \frac{i}{k}J';\\
P(\Omega) &= \frac{1}{k}(Q+K) \pm \frac{i}{k}(Q'+K'); & iP'(\Omega) &= \frac{i}{k}(Q'+K');\\
Q(\Omega) &= \frac{1}{k}(J-K) \pm \frac{i}{k}(J'-K'); & iQ'(\Omega) &= \frac{i}{k}(J'-K').
\end{aligned}$$

$$\text{V.}\quad \omega = \begin{vmatrix} \mp 1, & +1 \\ -1, & 0 \end{vmatrix} \times \Omega = -\frac{1}{\mp 1 + \Omega}.$$

$$\begin{aligned}
K(\Omega) &= kK'; & iK'(\Omega) &= ik[K \pm iK'];\\
J(\Omega) &= -\frac{Q'}{k}; & iJ'(\Omega) &= \frac{i}{k}[-Q \mp iQ'];\\
P(\Omega) &= -\frac{Q'+K'}{k}; & iP'(\Omega) &= \frac{i}{k}[-Q-K \mp i(Q'+K')];\\
Q(\Omega) &= -\frac{J'-K'}{k}; & iQ'(\Omega) &= \frac{i}{k}[-J+K \mp i(J'-K')].
\end{aligned}$$

$$\text{VI.}\quad \omega = \begin{vmatrix} 0, & 1 \\ -1, & \mp 1 \end{vmatrix} \times \Omega = \frac{-1 \mp \Omega}{\Omega}.$$

$K(\Omega) = k'(K' \mp iK)$; $\qquad iK'(\Omega) = ik'K$;

$J(\Omega) = \frac{1}{k'}[K' - J' \mp i(K-J)]$; $\qquad iJ'(\Omega) = \frac{i}{k'}(K-J)$;

$P(\Omega) = \frac{1}{k'}[K' - Q' \mp i(K-Q)]$; $\qquad iP'(\Omega) = \frac{i}{k'}(K-Q)$;

$Q(\Omega) = -\frac{1}{k'}(J' \mp iJ)$; $\qquad iQ'(\Omega) = -\frac{i}{k'}J.$

The nine typical transformations of the second order (see Art. 34) give the following formulæ for Q and Q'. They are derived by differentiation from the corresponding formulæ for K and K' in accordance with the equations (1), Note III, p. 1.

I.

(*a*) $$\omega = C_{1,1} \times \Omega = \begin{vmatrix} 1, & 1 \\ -1, & 1 \end{vmatrix} \times \Omega = \frac{\Omega - 1}{\Omega + 1}.$$

$$K(\Omega) = \eta^2 \sqrt{(kk')}\,(K - iK'),$$
$$K'(\Omega) = \eta^{-2} \sqrt{(kk')}\,(K + iK'),$$
$$Q(\Omega) = \frac{1}{i\sqrt{(2kk')}}\,[Q + (k^2 - \tfrac{1}{2})K],$$
$$Q'(\Omega) = \frac{1}{i\sqrt{(2kk')}}\,[Q' + (k^2 - \tfrac{1}{2})K'].$$

(*b*) $$\omega = C'_{1,1} \times \Omega = \begin{vmatrix} -1, & 1 \\ -1, & -1 \end{vmatrix} \times \Omega = \frac{1 + \Omega}{1 - \Omega}.$$

$$K(\Omega) = \eta^{-2} \sqrt{(kk')}\,(K + iK'),$$
$$K'(\Omega) = \eta^2 \sqrt{(kk')}\,(K - iK'),$$
$$Q(\Omega) = \frac{i}{\sqrt{(2kk')}}\,[Q' + (k^2 - \tfrac{1}{2})K'],$$
$$Q'(\Omega) = \frac{i}{\sqrt{(2kk')}}\,[Q + (k^2 - \tfrac{1}{2})K].$$

II.

$$\omega = C_{2,1} \times \Omega = \begin{vmatrix} 1, & 0 \\ 1, & 2 \end{vmatrix} \times \Omega = 1 + 2\Omega.$$

$$K\ (\Omega) = (k' - ik)\, K,$$

$$K'\,(\Omega) = \tfrac{1}{2}\,(k' - ik)\,(K' + iK),$$

$$Q\ \ (\Omega) = 2\,(k' + ik)\,[Q + ikk'K],$$

$$Q'\ (\Omega) = (k' + ik)\,[Q' + ikk'K' + i\,(Q + ikk'K)].$$

III.

$$\omega = C_{2,1} \times \Omega = \begin{vmatrix} 0, & 1 \\ -2, & 1 \end{vmatrix} \times \Omega = \frac{\Omega - 2}{\Omega}.$$

$$K\ (\Omega) = \tfrac{1}{2}\,(k' + ik)\,(K' + iK),$$

$$K'\,(\Omega) = (k' + ik)\, K,$$

$$Q\ \ (\Omega) = -\,(k' - ik)\,[Q' + ikk'K' + i\,(Q + ikk'K)],$$

$$Q'\ (\Omega) = -\,2\,(k' - ik)\,[Q + ikk'K].$$

IV.

$$\omega = C_{1,2} \times \Omega = \begin{vmatrix} 2, & 0 \\ -1, & 1 \end{vmatrix} \times \Omega = \frac{\Omega - 1}{2}.$$

$$K\ (\Omega) = \sqrt{k'}\, K,$$

$$K'\,(\Omega) = 2\,\sqrt{k'}\,(K' - \tfrac{1}{2}iK),$$

$$Q\ \ (\Omega) = \frac{1}{2\sqrt{k'}}\,[Q + \tfrac{1}{2}k^2K],$$

$$Q'\ (\Omega) = \frac{1}{2\sqrt{k'}}\,[2\,(Q' + \tfrac{1}{2}k^2K') - i\,(Q + \tfrac{1}{2}k^2K)].$$

V.

$$\omega = C_{3,2} \times \Omega = \begin{vmatrix} 0, & 2 \\ -1, & 0 \end{vmatrix} \times \Omega = -\frac{1}{2\Omega}.$$

$$K\ (\Omega) = (1 + k')\, K',$$

$$K'(\Omega) = \tfrac{1}{2}(1 + k')\, K,$$

$$Q\ (\Omega) = -\frac{2}{1+k'}\,[Q' + k'(1 - k')\, K'],$$

$$Q'\ (\Omega) = -\frac{1}{1+k'}\,[Q + k'(1 - k')\, K].$$

VI.

$$\omega = C_{3,2} \times \Omega = \begin{vmatrix} 2, & 0 \\ 0, & 1 \end{vmatrix} \times \Omega = \tfrac{1}{2}\Omega.$$

$$K\ (\Omega) = \tfrac{1}{2}(1 + k')\, K,$$

$$K'(\Omega) = (1 + k')\, K',$$

$$Q\ (\Omega) = \frac{1}{1+k'}\,[Q + k'(1 - k')\, K],$$

$$Q'\ (\Omega) = \frac{2}{1+k'}\,[Q' + k'(1 - k')\, K'].$$

VII.

$$\omega = C_{1,2} \times \Omega = \begin{vmatrix} -1, & 1 \\ -2, & 0 \end{vmatrix} \times \Omega = \frac{2}{1 - \Omega}.$$

$$K\ (\Omega) = \sqrt{k}\, K',$$

$$K'(\Omega) = 2\sqrt{k}\,[K - \tfrac{1}{2}iK'],$$

$$Q\ (\Omega) = -\frac{1}{2\sqrt{k}}(Q' - \tfrac{1}{2}k'^2 K'),$$

$$Q'\ (\Omega) = -\frac{1}{2\sqrt{k}}[2(Q - \tfrac{1}{2}k'^2 K) - i(Q' - \tfrac{1}{2}k'^2 K')].$$

VIII.

$$\omega = C_{2,2} \times \Omega = \begin{vmatrix} 1, & 0 \\ 0, & 2 \end{vmatrix} \times \Omega = 2\Omega.$$

$$K\ (\Omega) = (1+k)\,K,$$

$$K'(\Omega) = \tfrac{1}{2}(1+k)\,K',$$

$$Q\ (\Omega) = \frac{2}{1+k}[Q - k(1-k)\,K],$$

$$Q'\ (\Omega) = \frac{1}{1+k}[Q' - k(1-k)\,K'].$$

IX.

$$\omega = C_{2,2} \times \Omega = \begin{vmatrix} 0, & 1 \\ -2, & 0 \end{vmatrix} \times \Omega = -\frac{2}{\Omega}.$$

$$K\ (\Omega) = \tfrac{1}{2}(1+k)\,K',$$

$$K'(\Omega) = (1+k)\,K,$$

$$Q\ (\Omega) = -\frac{1}{1+k}[Q' - k(1-k)\,K'],$$

$$Q'\ (\Omega) = -\frac{2}{1+k}[Q - k(1-k)\,K].$$

Let $h(\omega) = i\dfrac{Q'(\omega)}{Q\ (\omega)}$; the formulæ of the Table show that if $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$ be a unit matrix of either of the types (1) or (ψ), the equation $\omega = \begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \times \Omega$ implies the equation

$$h(\omega) = \begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \times h(\Omega);$$

or, which is the same thing,

$$(1) \qquad \omega = 2a_1 - \cfrac{1}{2a_2 - \cfrac{1}{2a_3 \ldots - \cfrac{1}{2a_\nu - \cfrac{1}{\Omega}}}},$$

the quotients being even and the continued fraction subtractive; then also

$$(2) \qquad h(\omega) = 2a_1 - \frac{1}{2a_2 - \frac{1}{2a_3 \ldots - \frac{1}{2a_s - \frac{1}{h(\Omega)}}}}.$$

The property of the function $h(\omega)$ expressed by the equations (1) and (2), and the characteristic property of the elliptic function of the second species may be regarded as having a relation to one another comparable to that existing between the double periodicity of the elliptic functions of the first species and the quasi-periodicity of the modular functions. The plane on which the argument of the elliptic functions is represented is divided into elementary parallelograms, of which the sides are any two simultaneous elliptic periods; and the elliptic function of the first species has the same value at corresponding points in two different parallelograms, while the values of the function of the second species differ by a quantity of the same type as the difference of the vectors of the two points; e.g.

$$Z(x + 2mK + 2m'iK') = Z(x) + 2mQ + 2m'iQ'.$$

On the other hand, answering to the linear substitutions $\omega = \dfrac{c + d\Omega}{a + b\Omega}$, we have a division of the semi-plane on which ω is represented into lunular spaces equivalent to the reduced space (B) of fig. 2, referred to in Art. 38. At corresponding points in two such spaces the modular functions $\Phi(\omega)$ and $\Psi(\omega)$ have the same value, while the values of $h(\omega)$ are related to one another as the vectors of the two points.

VII.

Formulæ relating to the Elliptic Functions of the Second Species.

[This is to some extent equivalent to IV, but the two Notes could not be united together].

The Formula of Addition.

Let x_1, x_2, x_3 be three arguments of which the sum is zero; we have from the equation of Jacobi [Art. 10, (iii)].

$$\text{(i)} \qquad Z(x_1) + Z(x_2) + Z(x_3) = -\sum_{s=1}^{s=3} \frac{d}{dx_s} \log \vartheta_0\left(\frac{\pi x_s}{2K}\right).$$

In the equation (i) of Art. 6, let

$$\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu, \quad \nu_1 = \nu_2 = \nu_3 = \nu_4 = \nu, \quad x_1 + x_2 + x_3 = 0;$$

differentiate with regard to x_4, and write $x_4 = 0$ in the result. Representing for brevity $\vartheta'_1(0)\,\vartheta_1(x_1)\,\vartheta_1(x_2)\,\vartheta_1(x_3)$ by a_1, and

$$\vartheta_s(0)\,[\vartheta'_s(x_1)\,\vartheta_s(x_2)\,\vartheta_s(x_3) + \vartheta_s(x_1)\,\vartheta'_s(x_2)\,\vartheta_s(x_3) + \vartheta_s(x_1)\,\vartheta_s(x_2)\,\vartheta'_s(x_3)]$$

by a_s, ($s = 0, 2, 3$), we attribute successively to μ, ν the values $0, 0;\ 0, 1;\ 1, 0;$ we thus obtain the equations

$$a_1 + a_2 + a_3 + a_0 = 0,$$
$$a_1 + a_2 - a_3 - a_0 = 0,$$
$$a_1 - a_2 - a_3 + a_0 = 0,$$

or, which is the same thing,

$$a_1 = -a_2 = a_3 = -a_0.$$

Transforming the right-hand member of (i) by means of the equality $a_0 = -a_1$, we arrive at the equation of addition in its usual form; viz.,

$$\text{(ii)}\quad Z(x_1) + Z(x_2) + Z(x_3) = -k^2 \sin\operatorname{am} x_1 \sin\operatorname{am} x_2 \sin\operatorname{am} x_3;$$
$$x_1 + x_2 + x_3 = 0.$$

The Formulæ of Transformation.

In the equation of Jacobi [Art. 10 (iii)] we write $\dfrac{x}{M}$ for x, and Ω for ω; we thus obtain

$$\text{(iii)}\quad \frac{1}{M} Z\left[\frac{x}{M}, \lambda^2\right] = \frac{J(\Omega)}{K(\Omega)}\frac{x}{M^2} - \frac{d}{dx}\log\vartheta_0\left(\frac{\pi x}{2MK(\Omega)}, \Omega\right)$$
$$= \frac{J(\Omega)}{K(\Omega)}\frac{x}{M^2} - \frac{d}{dx}\log\vartheta_0\left(\frac{\pi x(a+b\Omega)}{2K}, \Omega\right),$$

where $\omega = \begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \times \Omega$, $\lambda = \phi^4(\Omega)$, and M is the multiplier corresponding to the transformation. For brevity, we suppose that the matrix $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$ is of the uneven determinant Δ, and satisfies the congruential conditions $b \equiv c \equiv 0$, $a \equiv 1$, mod 8, assumed to exist in Art. 33. Putting $h = 2K$ in equation (xii)

of that article, and designating by C' a quantity not containing x, we find

$$\vartheta_0\left[(a+b\Omega)\frac{\pi x}{2K},\ \Omega\right]=C'\times e^{-i\pi b(a+b\Omega)\frac{x^2}{4K^2}}\times\vartheta_0^{\Delta}\left(\frac{\pi x}{2K},\ \omega\right)$$
$$\times\prod_{j=1}^{j=\frac{1}{2}(\Delta-1)}\left[1-k^2\sin^2 4j\zeta\sin^2\mathrm{am}\,x\right],$$

or taking the logarithm, and differentiating

$$\frac{d}{dx}\log\vartheta_0\left[(a+b\Omega)\frac{\pi x}{2K},\ \Omega\right]$$
$$=-\frac{i\pi b\,(a+b\Omega)\,x}{2K^2}+\Delta\frac{d}{dx}\log\vartheta_0\left[\frac{\pi x}{2K},\ \omega\right]$$
$$+\sum_{j=1}^{j=\frac{1}{2}(\Delta-1)}\frac{d}{dx}\log\left[1-k^2\sin^2 4j\zeta\sin^2\mathrm{am}\,x\right],$$

whence finally

(iv) $$\frac{1}{M}Z\left(\frac{x}{M},\ \lambda^2\right)-\Delta Z(x,\ k^2)+2Cx$$
$$=2k^2\sin\mathrm{am}\,x\cos\mathrm{am}\,x\,\Delta\mathrm{am}\,x\sum_j\frac{\sin^2\mathrm{am}\,4j\zeta}{1-\sin^2\mathrm{am}\,4j\zeta\sin^2\mathrm{am}\,x},$$

the value of the constant C being

(v) $$\begin{cases}2C=\Delta\dfrac{J(\omega)}{K(\omega)}-\dfrac{1}{M^2}\dfrac{J(\Omega)}{K(\Omega)}-\dfrac{i\pi b\,(a+b\Omega)}{2K^2(\omega)}\\[2ex]\quad=\Delta\dfrac{J(\omega)}{K(\omega)}-\dfrac{1}{M^2}\dfrac{J(\Omega)}{K(\Omega)}-\dfrac{i\pi b}{2MK(\omega)\,K(\Omega)}.\end{cases}$$

Dividing equation (iv) by x, putting $x=0$, and observing that $\lim\dfrac{Z(x)}{x}=0$, we also have

(vi) $$C=\Sigma_j k^2\sin^2\mathrm{am}\,4j\zeta.$$

If we denote $\sin\mathrm{am}\,x$ by u, the value of $\sin\mathrm{am}\left(\dfrac{x}{M},\ \lambda\right)$, Art. 33, equation (xvii) is of the form

$$\frac{u}{M}\,\frac{1+A_1u^2+A_2u^4+\ldots+A_{\frac{1}{2}(\Delta-1)}u^{\Delta-1}}{1+B_1u^2+B_2u^4+\ldots+B_{\frac{1}{2}(\Delta-1)}u^{\Delta-1}}$$

$$=(-1)^{\frac{1}{2}(\Delta-1)}\frac{u^{\Delta}}{M.\Pi\sin^2\mathrm{am}\,4j\zeta}\times\frac{\chi\left(\frac{1}{k^2u^2}\right)}{\chi(u^2)}$$

$$=\frac{k^{\frac{1}{2}\Delta}u^{\Delta}}{\lambda^{\frac{1}{2}}}+\frac{\chi\left(\frac{1}{k^2u^2}\right)}{\chi(u^2)},$$

if
$$\chi(u^2) = 1 + B_1 u^2 + B_2 u^4 + \ldots$$
$$= \Pi\left[1 - k^2 \sin^2 \mathrm{am}\, 4j\zeta \sin^2 \mathrm{am}\, x\right].$$

Employing this notation, we have for the right-hand member of (i) the equivalent expression

$$-2 \cos \mathrm{am}\, x \Delta\, \mathrm{am}\, x \frac{B_1 u + 2B_2 u^3 + 3B_3 u^5 + \ldots}{1 + B_1 u^2 + B_2 u^4 + \ldots},$$

so that
$$C = -B_1.$$

The constant C is a rational function of k^2 and λ^2; the integrals $K(\omega)$, $K(\Omega)$, $J(\omega)$, $J(\Omega)$ entering into the expressions (iii) only apparently: the value is in fact

(vii)
$$2C = \Delta k^2 - \frac{\lambda^2}{M^2} - \frac{2\Delta k^2 k'^2}{M} \frac{dM}{d.k^2}.*$$

It may be observed that the coefficients B_2, B_3, ... like B_1, are all rational functions of k^2 and λ^2; for these coefficients are linear functions of the quantities $\Sigma k^{2s} \sin^{2s} \mathrm{am}\, 4j\zeta$, which are rational in k^2 and λ^2, as may be seen by expanding the elliptic functions in equation (ii) in series proceeding by powers of x, and equating coefficients of like powers of x. The same thing is true for the coefficients A, which are linear in the quantities $\Sigma \sin^{-2s} \mathrm{am}\, 4j\zeta$, and which are also connected with the coefficients B by the relation

$$A_s = \frac{k^{2s+1} M}{k^{\frac{1}{2}\Delta} \lambda^{\frac{1}{2}}} B_{\frac{1}{2}(\Delta-1)-s},$$

where the coefficient of B is rational in k^2 and λ^2, because $\Pi \sin^2 \mathrm{am}\, 4j\zeta$ is so.

The right-hand member of equation (i) may be transformed by means of the formula of addition; viz. this formula gives

$$Z(x+y) + Z(x-y) - 2Z(x)$$
$$= [Z(x+y) + Z(x-y) - Z(2x)] - [2Z(x) - Z(2x)]$$
$$= \frac{2k^2 \sin \mathrm{am}\, x \cos \mathrm{am}\, x\, \Delta\, \mathrm{am}\, x \sin^2 \mathrm{am}\, y}{1 - k^2 \sin^2 \mathrm{am}\, x \sin^2 \mathrm{am}\, y},$$

* [C is the same as the H of Note IV. (pp. 8–10). In equation (6) of that Note, $\frac{K(\Omega)}{J(\Omega)}$ should be $\frac{J(\Omega)}{K(\Omega)}$ and in two places 2 should be i.]

whence

$$\text{(viii)} \qquad \frac{1}{M} Z\left(\frac{x}{M}, \lambda\right) - \Delta Z(x, k) + 2Cx = \Sigma\, Z(x + 4j''\zeta, k),$$

the summation extending to every value of j'' from $-\frac{1}{2}(\Delta - 1)$ to $\frac{1}{2}(\Delta - 1)$. The formula (viii) may also be obtained by integrating the sum of the squares of the roots of equation (xxv), Art. 33.

To complete the preceding theory we add the following list of the transformations of $Z(x)$ by unit matrices.

I.

For any unit matrix $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \equiv \begin{vmatrix} 1, & 0 \\ 0, & 1 \end{vmatrix}$ (mod 2),

$$(-1)^{\frac{1}{2}(a-1)} Z[(-1)^{\frac{1}{2}(a-1)} x, k] = Z[x, k].$$

II.

$$\omega = \begin{vmatrix} 0, & 1 \\ -1, & 0 \end{vmatrix} \times \Omega = -\frac{1}{\Omega},$$

$$iZ[ix, k'] = Z(x, k) - \left[x - \frac{\sin \operatorname{am} x \Delta \operatorname{am} x}{\cos \operatorname{am} x}\right].$$

III.

$$\omega = \begin{vmatrix} 1, & 0 \\ \pm 1, & 1 \end{vmatrix} \times \Omega = \Omega \pm 1,$$

$$k' Z\left[k'x, \mp \frac{ik}{k'}\right] = Z(x, k) - k^2 \left[x - \frac{\sin \operatorname{am} x \cos \operatorname{am} x}{\Delta \operatorname{am} x}\right].$$

IV.

$$\omega = \begin{vmatrix} \pm 1, & 1 \\ 0, & \pm 1 \end{vmatrix} \times \Omega = \frac{\Omega}{1 \pm \Omega},$$

$$kZ\left[kx, \frac{1}{k}\right] = Z[x, k].$$

V.

$$\omega = \begin{vmatrix} \pm 1, & 1 \\ -1, & 0 \end{vmatrix} \times \Omega = -\frac{1}{\Omega \pm 1},$$

$$ikZ\left[ikx, \mp\frac{ik'}{k}\right] = Z[x, k] - k^2\left(x - \frac{\sin\operatorname{am}x \cos\operatorname{am}x}{\Delta\operatorname{am}x}\right).$$

VI.

$$\omega = \begin{vmatrix} 0, & 1 \\ -1, & \pm 1 \end{vmatrix} \times \Omega = -\frac{1 \mp \Omega}{\Omega},$$

$$ik'Z\left[ik'x, \frac{1}{k'}\right] = Z[x, k] - \left(x - \frac{\sin\operatorname{am}x \,\Delta\operatorname{am}x}{\cos\operatorname{am}x}\right).$$

These formulæ may be verified by transforming the function $\sin\operatorname{am}x$ in the equation $Z(x) = \int_0^x k^2 \sin^2\operatorname{am}x\,dx$. For example,

$iZ(ix, k') - Z(x, k) + x$

$$= i\int_0^{ix} k'^2 \sin^2\operatorname{am}(x, k')\,dx - \int_0^x k^2 \sin^2\operatorname{am}(x, k)\,dx + x$$

$$= -\int_0^x [k'^2 \sin^2\operatorname{am}(ix, k') + k^2 \sin^2\operatorname{am}(x, k) - 1]\,dx$$

$$= \int_0^x \left[k'^2 \frac{\sin^2\operatorname{am}(x, k)}{\cos^2\operatorname{am}(x, k)} + \Delta^2\operatorname{am}(x, k)\right] dx$$

$$= \frac{\sin\operatorname{am}x \,\Delta\operatorname{am}x}{\cos\operatorname{am}x}.$$

The formula IV. (which is an immediate consequence of the corresponding formula for $\sin\operatorname{am}x$) changes II. into VI., III. into V., and *vice versâ.* The formulæ may also be deduced from the equation (iii) *suprà*, but the elimination of the complete integrals requires the use of some additional formulæ.

Taking again as an example the formula II., we have

$$iZ(ix, k') = -\frac{J(\Omega)}{K(\Omega)}x - \frac{d}{dx}\log\vartheta_0\left(\frac{\pi x\Omega}{2K}, \Omega\right)$$

$$= -\frac{J(\Omega)}{K(\Omega)}x + \frac{i\pi\Omega x}{2K^2} - \frac{d}{dx}\log\vartheta_2\left(\frac{\pi x}{2K}, \omega\right)$$

$$= -\frac{J(\Omega)}{K(\Omega)}x + \frac{i\pi\Omega x}{2K^2} - \frac{d}{dx}\log\vartheta_0\left(\frac{\pi x}{2K}, \varpi\right)$$

$$- \frac{d}{dx}\log\cos\mathrm{am}(x, k)$$

$$= \quad Z(x, k) - x\left[\frac{J(\Omega)}{K(\Omega)} + \frac{J(\omega)}{K(\omega)} - \frac{i\pi\Omega}{2K^2}\right]$$

$$- \frac{d}{dx}\log\cos\mathrm{am}(x, k).$$

But $\quad J(\Omega) = K'(\omega) - J'(\omega), \quad K(\Omega) = K'(\omega),$

$$\Omega = \frac{iK(\omega)}{K'(\omega)};$$

and the coefficient of $-x$ becomes

$$1 + \frac{K'J - KJ' + \frac{1}{2}\pi}{KK'}, \text{ that is, } 1,$$

in accordance with the formula II.

Lastly, for the three typical quadratic transformations, writing for brevity s, c, d to denote $\sin\mathrm{am}(x, k)$, $\cos\mathrm{am}(x, k)$, $\Delta\mathrm{am}(x, k)$ respectively, we have the formulæ

(1) $$\omega = \begin{vmatrix} 1, & 0 \\ 1, & 2 \end{vmatrix} \times \Omega$$

$$(k' - ik)\, Z\left[(k' - ik)\,x,\ \frac{2\sqrt{(ikk')}}{k + ik'}\right] - 2Z[x, k]$$

$$= -2k(k + ik')\left[x - \frac{csd}{1 - k(k + ik')s^2}\right];$$

(2) $$(1 + k')\, Z\left[(1 + k')\,x,\ \frac{1 - k'}{1 + k'}\right] - 2Z[x, k]$$

$$= -k^2\left[x - \frac{sc}{d}\right];$$

(3) $$(1 + k)\, Z\left[(1 + k)\,x,\ \frac{2\sqrt{k}}{1 + k}\right] - 2Z[x, k]$$

$$= 2k\left[x - \frac{scd}{1 + ks^2}\right];$$

which may be verified by either of the methods exemplified

in the case of the linear transformation $\omega = -\dfrac{1}{\Omega}$. Observing that

$$2Z(x, k) - Z[2x, k] = -k^2 \sin^2 \mathrm{am}\, x \sin \mathrm{am}\, 2x,$$

we may write the equation (3) in the form

$$(1+k)\, Z\left[(1+k)\,x,\ \frac{2\sqrt{k}}{1+k}\right] - Z[2x, k] = k\,[2x - \sin \mathrm{am}\, 2x].$$

Employing the notation of Legendre,

$$F(\phi, k) = \int_0^\phi \frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}},\quad E(\phi, k) = \int_0^\phi \sqrt{(1-k^2\sin^2\phi)}\,d\phi;$$

and writing

$$\phi = \mathrm{am}\,[2x, k],\quad \psi = \mathrm{am}\left[(1+k)x,\ \frac{2\sqrt{k}}{1+k}\right],$$

so that $\qquad Z[2x, k] = F(\phi, k) - E(\phi, k),$

$$Z\left[(1+k)x,\ \frac{2\sqrt{k}}{1+k}\right] = F\left(\psi,\ \frac{2\sqrt{k}}{1+k}\right) - E\left(\psi,\ \frac{2\sqrt{k}}{1+k}\right)$$

$$= \frac{1+k}{2}\, F(\phi, k) - E\left(\psi,\ \frac{2\sqrt{k}}{1+k}\right),$$

we find

$$(4)\quad \begin{cases} \tfrac{1}{2}k'^2 F(\phi, k) = E(\phi, k) - (1+k)\,E\left(\psi,\ \dfrac{2\sqrt{k}}{1+k}\right) + k\sin\phi, \\ \sin(2\psi - \phi) = k\sin\phi, \end{cases}$$

a celebrated formula, due to Landen, which serves to express an elliptic integral of the first species having a real modulus less than unity by means of two elliptic integrals of the second species having modules of the same character.

A similar, and indeed equivalent, formula is obtained from the equation (2); viz. writing

$$\phi = \mathrm{am}\,[x, k],\quad \psi = \mathrm{am}\left[(1+k')x,\ \frac{1-k'}{1+k'}\right],$$

we find

$$(5)\quad \begin{cases} k' F(\phi, k) = \tfrac{1}{2}(1+k')\,E\left(\psi,\ \dfrac{1-k'}{1+k'}\right) - E(\phi, k) \\ \qquad\qquad + \tfrac{1}{2}(1-k')\sin\psi, \\ \tan(\psi - \phi) = \tan\phi. \end{cases}$$

VIII.

The Functions $Al(x)$ of Weierstrass.

The Abelian functions of M. Weierstrass are defined by the equations $[s=0, 1, 2, 3]$,

$$\text{(i)} \qquad Al_s(x) = e^{-\frac{J}{2K}x^2} \frac{\vartheta_s\left(\frac{\pi x}{2K}\right)}{a_s},$$

where, if $s=1$,

$$a_s = \lim_{x=0} \left[\frac{\vartheta_1\left(\frac{\pi x}{2K}\right)}{x}\right];$$

and, if $s=0, 2, 3$,

$$a_s = \vartheta_s(0),$$

and where

$$J = \int_0^x k^2 \sin^2 \mathrm{am}\, x\, dx.$$

These functions which differ from the corresponding Theta functions of $\frac{\pi x}{2K}$ only by the presence of the exponential factor $e^{-\frac{J}{2K}x^2}$, and of a factor not containing x, possess, as is well known, the remarkable property that they can be developed in series proceeding by powers of x, of which the coefficients are integral functions of k^2 with integral coefficients, and which are convergent for all values of x. For our present purpose, there is a slight convenience in considering, instead of the functions $Al(x)$, multiples of these functions by the exponential $e^{\frac{1}{2}k^2x^2}$. These multiples we shall call, in what follows, Abelian functions, and we shall denote them by the symbols $A_0(x)$, $A_1(x)$, $A_2(x)$, $A_3(x)$, so that *e.g.*

$$\text{(ii)} \qquad A_0(x) = e^{-\frac{Q}{2K}x^2} \times \frac{\vartheta_0\left(\frac{\pi x}{2K}\right)}{\vartheta_0(0)}.$$

These functions respectively satisfy the partial differential equations

$$\text{(iii)} \qquad \frac{d^2 A_s}{dx^2} + 4k^2(1-k^2)\frac{dA_s}{d.k^2} - k^2(1-k^2)x^2 A_s + g_s A_s = 0,$$

where $g_0 = -k^2$, $g_1 = 1-2k^2$, $g_2 = 1-k^2$, $g_3 = 0$;
or, which is the same thing, the four functions $A_0\sqrt{k'}$, $A_1\sqrt{(kk')}$, $A_2\sqrt{k}$, A_3 all satisfy one and the same equation

$$\text{(iv)} \qquad \frac{d^2 A}{dx^2} + 4k^2(1-k^2)\frac{dA}{d.k^2} - k^2(1-k^2)x^2 A = 0.$$

The equations (iii) are a little simpler than the corresponding equations satisfied by the functions of M. Weierstrass, viz.

$$\text{(v)}\quad \frac{d^2 Al}{dx^2} + 2k^2 x \frac{dAl}{dx} + 4k^2(1-k^2)\frac{dAl}{d.k^2} + k^2x^2 Al + g_s Al = 0,$$

where $\quad g_0 = 0, \quad g_1 = 1 - k^2, \quad g_2 = 1, \quad g_3 = k^2;$

but the gain in simplicity is apparent rather than real, as the determination of the coefficients is not more easily affected in the functions A_s than in the functions Al_s. The equations (iii) or (iv), as well as (v) may be derived directly from the partial differential equation of the Theta functions

$$\frac{d^2\vartheta(x, \omega)}{dx^2} = \frac{4i}{\pi}\frac{d\vartheta(x, \omega)}{d\omega},$$

which gives

$$\frac{d\vartheta\left(\frac{\pi x}{2K}\right)}{d\omega} = -\frac{x}{K}\frac{d\vartheta\left(\frac{\pi x}{2K}\right)}{dx}\frac{dK}{d\omega} + \frac{K^2}{i\pi}\frac{d^2\vartheta\left(\frac{\pi x}{2K}\right)}{dx^2},$$

and in which we are to put

$$\vartheta_s\left(\frac{\pi x}{2K}\right) = e^{\frac{Q}{2K}x^2} \times A_s(x) \times \vartheta_s(0),$$

and to eliminate $d\omega$ by means of the equation

$$\frac{d.k^2}{d\omega} = -\frac{4}{i\pi}k^2(1-k^2)K^2,$$

the differentiations with regard to k^2 being effected by means of the formulæ

$$\frac{dK}{d.k^2} = -\frac{Q}{2k^2k'^2}, \quad \frac{dQ}{d.k^2} = -\tfrac{1}{2}K,$$

The Abelian functions are connected by a relation which may be inferred from the corresponding relation between the Theta functions; viz., using for a moment a notation with double suffixes,

$$A_{0,1} = A_0, \quad A_{1,1} = A_1, \quad A_{1,0} = A_2, \quad A_{0,0} = A_3,$$

and supposing $A_{m,n} = A_{\mu,\nu}$, if $m \equiv \mu$, $n \equiv \nu$ (mod 2), then from the formula

$$\vartheta_{m+\mu,\, n+\nu}(x) = e^{\mu i\pi\left[\frac{x}{\pi} + \frac{1}{4}\mu\omega + \frac{1}{2}(n+\nu)\right]} \times \vartheta_{m,n}(x + \tfrac{1}{2}\mu\omega\pi + \tfrac{1}{2}\nu\pi)$$

we obtain the following

$$A_{m,n}(x+\nu K+i\mu K')$$
$$=e^{(\nu Q+i\mu Q')(x+\frac{1}{2}\nu K+\frac{1}{2}i\mu K')}\times e^{-\frac{1}{4}i\pi\mu(2n+\nu)}\times\xi\times A_{m+\mu,\,n+\nu}(x),$$

where
$$\xi=\frac{a_{m+\mu,\,n+\nu}}{a_{m,n}}.$$

General formulæ for the transformation of the Abelian functions are immediately deducible from the formulæ for the transformation of the Theta functions.

Thus, since in general, whatever be the transformation $\omega=\dfrac{c+d\Omega}{a+b\Omega}$, we have

$$\vartheta_{\mu,\nu}\left[(\alpha+\beta\Omega)\frac{\pi x}{2K},\Omega\right]=e^{-i\pi\beta(\alpha+\beta)\Omega\frac{x^2}{4K^2}}\times T,$$

where T is a homogeneous function of order n of two of the Theta functions $\vartheta\left(\dfrac{\pi x}{h},\omega\right)$, then, attending to the equation

$$\frac{1}{M^2}\frac{Q(\Omega)}{2K(\Omega)}-n\frac{Q(\omega)}{2K(\omega)}-\frac{i\pi\beta(\alpha+\beta\Omega)}{4K^2(\omega)}$$
$$=-nk^2(1-k^2)\frac{1}{M}\frac{dM}{d.k^2},$$

we find

$$A_s\left(\frac{x}{M},\lambda^2\right)=e^{-nk^2(1-k^2)\frac{1}{M}\frac{dM}{d.k^2}x^2}\times T_s,$$

where T_s is a homogeneous function of the order n of two of the functions $A(x,k^2)$.

In particular, if n is uneven and the transformation is primary, we have

$$A_s\left(\frac{x}{M},\lambda^2\right)=e^{-nk^2(1-k^2)\frac{1}{M}\frac{dM}{d.k^2}x^2}\times A_0^{\,n}(x,k^2)\times U_s,$$

where U_s may be expressed in terms of the Abelian functions; *e.g.*

$$U_3=\Delta\,\mathrm{am}\,x\,\Pi\left[1-k^2\sin^2\mathrm{coam}\,4j\eta\,\sin^2\mathrm{am}\,x\right]$$
$$=\frac{A_3(x)}{A_0(x)}\,\Pi\left[1-k^2\frac{A_2^{\,2}(4j\eta)}{A_3^{\,2}(4j\eta)}\frac{A_1^{\,2}(x)}{A_0^{\,2}(x)}\right].$$

For the linear transformations of the Abelian functions we find

I. If $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \equiv \begin{vmatrix} 1, & 0 \\ 0, & 1 \end{vmatrix} \bmod 2,$

$$\lambda^2 = k^2, \quad \frac{1}{M} = (-1)^{\frac{1}{2}(a-1)},$$

$$A_s = A\,.$$

II. If $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \equiv \begin{vmatrix} 0, & 1 \\ -1, & 0 \end{vmatrix} \bmod 2,$

$$\lambda^2 = 1 - k^2, \quad \frac{1}{M} = (-1)^{\frac{1}{2}(b-1)} i,$$

$$A_1(ix,\, 1-k^2) = iA_1(x,\, k^2),$$
$$A_2(ix,\, 1-k^2) = A_0(x,\, k^2),$$
$$A_3(ix,\, 1-k^2) = A_3(x,\, k^2),$$
$$A_0(ix,\, 1-k^2) = A_2(x,\, k^2).$$

These equations signify that the coefficients of x^{4n+1} and x^{4n} in A_1 and A_3 respectively are functions of $k^2(1-k^2)$, and the coefficients of x^{4n+3} and x^{4n+2} are functions of $k^2(1-k^2)$ multiplied by $1-2k^2$; while corresponding coefficients in A_0 and A_2 are of the form $p \pm (1-2k^2)\,q$ or $\pm p + (1-2k^2)\,q$, according as the exponent of x is evenly or unevenly even.

III. If $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \equiv \begin{vmatrix} 1, & 0 \\ 1, & 1 \end{vmatrix} \bmod 2, \quad \lambda^2 = \frac{k^2}{k^2-1},$

$$\frac{1}{M} = k', \quad k^2(1-k^2)\,\frac{d \log M}{d.k^2} = \tfrac{1}{2}k^2.$$

$$A_1\left(k'x,\, \frac{k^2}{k^2-1}\right) = k' e^{-\frac{1}{2}k^2x^2} A_1(x,\, k^2),$$

$$A_2\left(k'x,\, \frac{k^2}{k^2-1}\right) = e^{-\frac{1}{2}k^2x^2} A_2(x,\, k^2),$$

$$A_3\left(k'x,\, \frac{k^2}{k^2-1}\right) = e^{-\frac{1}{2}k^2x^2} A_0(x,\, k^2),$$

$$A_0\left(k'x,\, \frac{k^2}{k^2-1}\right) = e^{-\frac{1}{2}k^2x^2} A_3(x,\, k^2),$$

IV. If $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} = \begin{vmatrix} 1, & 1 \\ 0, & 1 \end{vmatrix}$ mod 2, $\lambda^2 = \dfrac{1}{k^2}$,

$$\frac{1}{M} = k, \quad k^2(1-k^2)\frac{d\log M}{d.k^2} = \tfrac{1}{2}(1-k^2),$$

$$A_1\left(kx, \frac{1}{k^2}\right) = ke^{-\frac{1}{2}k'^2x^2}A_1(x, k^2),$$

$$A_2\left(kx, \frac{1}{k^2}\right) = e^{-\frac{1}{2}k'^2x^2}A_3(x, k^2),$$

$$A_3\left(kx, \frac{1}{k^2}\right) = e^{-\frac{1}{2}k'^2x^2}A_2(x, k^2),$$

$$A_0\left(kx, \frac{1}{k^2}\right) = e^{-\frac{1}{2}k'^2x^2}A_0(x, k^2),$$

V. If $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \equiv \begin{vmatrix} 1, & 1 \\ -1, & 0 \end{vmatrix}$ mod 2, $\lambda^2 = \dfrac{1}{1-k^2}$,

$$\frac{1}{M} = ik, \quad k^2(1-k^2)\frac{d\log M}{d.k^2} = \tfrac{1}{2}(1-k^2),$$

$$A_1\left(ikx, \frac{1}{1-k^2}\right) = ike^{-\frac{1}{2}k'^2x^2}A_1(x, k^2),$$

$$A_2\left(ikx, \frac{1}{1-k^2}\right) = e^{-\frac{1}{2}k'^2x^2}A_0(x, k^2),$$

$$A_3\left(ikx, \frac{1}{1-k^2}\right) = e^{-\frac{1}{2}k'^2x}A_2(x, k^2),$$

$$A_0\left(ikx, \frac{1}{1-k^2}\right) = e^{-\frac{1}{2}k'^2x^2}A_3(x, k^2).$$

VI. If $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \equiv \begin{vmatrix} 0, & 1 \\ -1, & -1 \end{vmatrix}$ mod 2, $\lambda^2 = 1 - \dfrac{1}{k^2}$,

$$\frac{1}{M} = ik', \quad k^2(1-k^2)\frac{d\log M}{d.k^2} = \tfrac{1}{2}k^2,$$

$$A_1\left(ik'x, 1-\frac{1}{k^2}\right) = ik'e^{-\frac{1}{2}k^2x^2}A_1(x, k^2),$$

$$A_2\left(ik'x, 1-\frac{1}{k^2}\right) = e^{-\frac{1}{2}k^2x^2}A_3(x, k^2),$$

$$A_3\left(ik'x, 1-\frac{1}{k^2}\right) = e^{-\frac{1}{2}k^2x^2}A_0(x, k^2),$$

$$A_0\left(ik'x, 1-\frac{1}{k^2}\right) = e^{-\frac{1}{2}k^2x^2}A_2(x, k^2).$$

The functions $\Sigma = \frac{1}{\sqrt{M}} e^{hx^2} \bar{A}\left(\frac{x}{M}, \lambda^2\right)$ satisfy a partial differential equations with respect to x and k^2. We have from that equation, if $A = \bar{A}\left(\frac{x}{M}, \lambda^2\right)$,

$$M^2 \frac{d^2 A}{dx^2} + 4\lambda^2 (1-\lambda^2) \left[\frac{dA}{d.\lambda^2}\right] - \frac{\lambda^2(1-\lambda^2)}{M^2} x^2 A = 0,$$

where $$\left[\frac{dA}{d.\lambda^2}\right] = \frac{dA}{d.\lambda^2} - Mx \frac{dA}{dx} \cdot \frac{d\left(\frac{1}{M}\right)}{d.\lambda^2}.$$

Observing that

$$\frac{\lambda^2(1-\lambda^2)}{M^2} \frac{d}{d.\lambda^2} = nk^2 (1-k^2) \frac{d}{d.k^2},$$

we find $$\frac{d^2 A}{dx^2} + 4nk^2 (1-k^2) \frac{1}{M} \frac{dM}{d.k^2} . x \frac{dA}{dx}$$

$$+ 4nk^2 (1-k^2) \frac{dA}{d.k^2} - \frac{\lambda^2(1-\lambda^2)}{M^4} x^2 A = 0;$$

or, substituting for A its value $\sqrt{M} e^{-hx^2} \Sigma$, and dividing by $e^{-hx^2} \sqrt{M}$,

$$\frac{d^2\Sigma}{dx^2} + 4nk^2 (1-k^2) \frac{d\Sigma}{d.k^2}$$

$$- x^2 \left[\frac{\lambda^2(1-\lambda^2)}{M^4} + 4h^2 + 4nk^2 (1-k^2) \frac{dh}{d.k^2}\right] \Sigma = 0.$$

But the coefficient of $-x^2\Sigma$ in this equation is equal to $n^2k^2(1-k^2)$. Hence, we have finally

$$\frac{d^2\Sigma}{dx^2} + 4nk^2 (1-k^2) \frac{d\Sigma}{d.k^2} - n^2k^2 (1-k^2) x^2 \Sigma = 0,$$

which is satisfied by the four functions

$$\sqrt{\left(\frac{\lambda'}{M}\right)} e^{hx^2} A_0\left(\frac{x}{M}, \lambda^2\right), \quad \sqrt{\left(\frac{\lambda\lambda'}{M}\right)} e^{hx^2} A_1\left(\frac{x}{M}, \lambda^2\right),$$

$$\sqrt{\left(\frac{\lambda}{M}\right)} e^{hx^2} A_2\left(\frac{x}{M}, \lambda^2\right), \quad \sqrt{\left(\frac{1}{M}\right)} e^{hx^2} A_3\left(\frac{x}{M}, \lambda^2\right).$$

The determinant of the transformation n may have any positive integral value, and the constituents of the matrix of transformation may have any integral values.

If we replace x by $\dfrac{x}{\sqrt{n}}$ and $\dfrac{1}{M}$ by $\dfrac{\sqrt{n}}{\mu}$, we see that the functions

$$\sqrt{\mu}\times\overline{A}_s\left(\frac{x}{\mu},\ \lambda^2\right)\times e^{k^2(1-k^2)\frac{d\log M}{d.k^2}x^2}$$

satisfy an equation of the same form as that satisfied by $\overline{A}_s(x, k^2)$.

We next proceed to form the equation satisfied by

$$\sigma=\Sigma\div\overline{A}^n(x, k^2).$$

Substituting for Σ and dividing by A^n, and observing that

$$\frac{d\Sigma}{dx}=A^n\frac{d\sigma}{dx}+nA^{n-1}\frac{dA}{dx}\sigma,$$

$$\frac{d^2\Sigma}{dx^2}=A^n\frac{d^2\sigma}{dx^2}+2nA^{n-1}\frac{dA}{dx}\frac{d\sigma}{dx}+n(n-1)A^{n-2}\left(\frac{dA}{dx}\right)^2\sigma$$
$$+nA^{n-1}\frac{d^2A}{dx^2}\sigma,$$

$$\frac{d\Sigma}{d.k^2}=A^n\frac{d\sigma}{d.k^2}+nA^{n-1}\frac{dA}{d.k^2}\sigma,$$

we find

$$\frac{d^2\sigma}{dx^2}+2n\frac{1}{A}\frac{dA}{dx}\frac{d\sigma}{dx}+4nk^2(1-k^2)\frac{d\sigma}{d.k^2}$$
$$+n\sigma\left[(n-1)\left(\frac{1}{A}\frac{dA}{dx}\right)^2+\frac{1}{A}\frac{d^2A}{dx^2}\right.$$
$$\left.+4nk^2(1-k^2)\frac{1}{A}\frac{dA}{d.k^2}-nk^2(1-k^2)x^2\right]=0;$$

or, after all reductions,

$$\frac{d^2\sigma}{dx^2}+2nZ_0\frac{d\sigma}{dx}+4nk^2(1-k^2)\frac{d\sigma}{d.k^2}-n(n-1)k^2\,\text{cn}^2x.\sigma=0.$$

Lastly, if in this equation we introduce $s = \sin \operatorname{am} x$ and k^2, as the independent variables instead of x and k^2; we have

$$\frac{d\sigma}{dx} = \frac{d\sigma}{ds} \cdot \frac{ds}{dx} = cd\frac{d\sigma}{ds},$$

$$\frac{d^2\sigma}{dx^2} = c^2d^2\frac{d^2\sigma}{ds^2} - s\frac{d\sigma}{ds}(d^2 + k^2c^2),$$

$$\left[\frac{d\sigma}{d.k^2}\right] = \frac{d\sigma}{d.k^2} + \frac{d\sigma}{ds} \cdot \frac{ds}{d.k^2} = \frac{d\sigma}{d.k^2} + \frac{d\sigma}{ds}\frac{1}{2k^2k'^2}[k^2sc^2 - cdZ_0];$$

where $\left[\frac{d\sigma}{d.k^2}\right]$ is the differential coefficient of σ taken on the supposition that σ is a function of the independent variables k^2 and x, and $\frac{d\sigma}{d.k^2}$ is the differential coefficient taken on the supposition that σ is a function of k^2 and s, and that k^2 does not vary in s, and we obtain finally

$$(a) \quad (1 - s^2)(1 - k^2s^2)\frac{d^2\sigma}{ds^2} + [(2n - 1)k^2 - 1 - 2(n-1)k^2s^2]s\frac{d\sigma}{ds}$$
$$- n(n - 1)k^2(1 - s^2)\sigma + 4nk^2(1 - k^2)\frac{d\sigma}{d.k^2} = 0,$$

in which equation the coefficient of $\frac{d\sigma}{ds}$ may be written in either of the two forms $(2n - 1)k^2c^2 - d^2$ and $2(n - 1)k^2c^2 - k'^2$.

The equation is satisfied by

$$\sqrt{\left(\frac{\lambda'}{Mk'^n}\right)}U_0, \quad \sqrt{\left(\frac{\lambda\lambda'}{Mk'^n}\right)}U_1, \quad \sqrt{\left(\frac{\lambda}{Mk'^n}\right)}U_2, \quad \sqrt{\left(\frac{1}{Mk'^n}\right)}U_3.$$

For our immediate purpose we add the term $n^2k^2\sigma$ to this equation in order to obtain the equation satisfied by

$$\sqrt{\left(\frac{\lambda'}{M}\right)}U_0, \quad \sqrt{\left(\frac{\lambda\lambda'}{M}\right)}U_1, \quad \sqrt{\left(\frac{\lambda}{M}\right)}U_2, \quad \sqrt{\left(\frac{1}{M}\right)}U_3.$$

There is some difficulty in applying the equation

$$\frac{d^2\Sigma}{dx^2} + 4nk^2(1 - k^2)\frac{d\Sigma}{d.k^2} - n^2k^2(1 - k^2)x^2\Sigma = 0$$

to the actual determination of the functions U_0, U_1, U_2, U_3. The following method serves to exhibit these functions in what may be termed their "canonical form."

Let the operating symbol

$$\frac{d^2}{dx^2} + 4nk^2(1 - k^2)\frac{d}{d.k^2} - n^2k^2(1 - k^2)x^2$$

of this equation be denoted by ∇; write also for brevity

$$4k^2(1 - k^2)\frac{d}{d.k^2} = \delta,$$

and, considering first the three even functions $e^{hx^2}\bar{A}_s\left(\frac{x}{M}, \lambda^2\right)$, let $\nu = \sqrt{\left(\frac{\lambda'}{M}\right)}$, $\sqrt{\left(\frac{\lambda}{M}\right)}$, $\sqrt{\left(\frac{1}{M}\right)}$ in the three cases $s=0$, 2, 3 respectively. We find, by expansion,

$$\sqrt{\left(\frac{\lambda'}{M}\right)}\, e^{hx^2} A_0\left(\frac{x}{M}, \lambda^2\right) = \sqrt{\left(\frac{\lambda'}{M}\right)}\left[1 + \left(2h + \frac{\lambda^2}{M^2}\right)\frac{x^2}{1.2} + \ldots\right],$$

$$\sqrt{\left(\frac{\lambda}{M}\right)}\, e^{hx^2} A_2\left(\frac{x}{M}, \lambda^2\right) = \sqrt{\left(\frac{\lambda}{M}\right)}\left[1 + \left(2h - \frac{1-\lambda^2}{M^2}\right)\frac{x^2}{1.2} + \ldots\right],$$

$$\sqrt{\left(\frac{1}{M}\right)}\, e^{hx^2} A_3\left(\frac{x}{M}; \lambda^2\right) = \sqrt{\left(\frac{1}{M}\right)}\left[1 + 2h\frac{x^2}{1.2} + \ldots\right];$$

so that we have in the three cases alike

$$\nu e^{hx^2} A\left(\frac{x}{M}, \lambda^2\right) = \nu - n\delta\nu\, \frac{x^2}{1.2} + \ldots$$

Let
$$\phi_0 = 1 + a_2\frac{x^4}{4!} - a_3\frac{x^6}{6!} + \ldots$$

be a series satisfying the equation $\nabla\phi_0 = 0$, so that the coefficients a_2, $a_3 \ldots$ are rational and integral functions of k^2 with integral coefficients; similarly, let

$$\phi_1 = \frac{x^2}{2!} + b_3\frac{x^6}{6!} - b_4\frac{x^8}{8!} + \ldots$$

be a function determined by the equation

$$\nabla\phi_1 = \phi_0;$$

let
$$\phi_2 = \frac{x^4}{4!} + c_4\frac{x^8}{8!} - c_5\frac{x^{10}}{10!} + \ldots$$

be determined by the equation

$$\nabla\phi_2 = \phi_1;$$

and so on continually. It will be found that these determinations are always possible; and that, as indicated in the case of the functions ϕ_0, ϕ_1, ϕ_2, the first s coefficients in the function ϕ_s are zero, and the values of the two following coefficients are unity and zero. All the coefficients are rational and integral in k^2, and the highest power of k^2

in the coefficient of $\frac{x^{2s+2s'}}{(2s+2s')!}$ in ϕ_s is s'. We then have the theorem

$$\nu e^{hx^2} A\left(\frac{x}{M}, \lambda^2\right) = \nu\phi_0 - n\delta\nu.\phi_1 + n^2\delta^2\nu.\phi_2 - n^3\delta^3\nu.\phi_3 + \ldots$$

For if Φ represent the series on the right-hand side, we have, evidently,

$$\nabla\Phi = \nu\nabla\phi_0 - (\nabla\phi_1 - \phi_0)\, n\delta\nu + (\nabla\phi_2 - \phi_1)\, n^2\delta^2\nu - \ldots;$$

i.e. $$\nabla\Phi = 0; \text{ also } \Phi = \nu - n\delta\nu.\frac{x^2}{1.2} + \ldots;$$

so that the series Φ satisfies the same partial differential equation as the function $\nu e^{hx^2} A\left(\frac{x}{M}, \lambda^2\right)$, while the first two terms of the series coincide with the first two terms of the expansion of the function; and this establishes the theorem, because, given the first two terms of the expansion, it can be continued by means of the partial differential equation in one way only.

We next denote the operating factor

$$\frac{d^2}{dx^2} + 2nZ_0(x)\frac{d}{dx} + 4nk^2(1-k^2)\frac{d}{d.k^2} - n(n-1)k^2\,\mathrm{cn}^2 x + n^2k^2$$

by D; and we observe that we have not only

$$D\left[\frac{\Phi}{A_0^n(x, k^2)}\right] = 0,$$

but also, separately,

$$D\left[\frac{\phi_0}{A_0^n}\right] = 0, \quad D\left[\frac{\phi_1}{A_0^n}\right] = \frac{\phi_0}{A_0^n}, \quad D\left[\frac{\phi_2}{A_0^n}\right] = \frac{\phi_1}{A_0^n}, \quad \ldots;$$

viz., each of these equations is derived from the corresponding equation $\nabla\phi_s = \phi_{s-1}$ precisely in the same way in which the equation $D\sigma = 0$ is derived from the equation $\nabla\Sigma = 0$.

We now expand the functions $\frac{\phi_s}{A_0^n}$ in series proceeding by powers of s; we observe that if $\psi_s(s)$ be this expansion, it follows from the properties of the functions ϕ_s that the first

term in $\psi_2(s)$ is $\frac{s^{2r}}{2r\,!}$, and that

$$\psi_2(s)=\frac{s^{2r}}{(2r)\,!}+\rho_1\frac{s^{2r+2}}{(2r+2)\,!}+\rho_2\frac{s^{2r+4}}{(2r+4)\,!}+\ldots,$$

where ρ_m is a rational and integral function of k^2 with integral coefficients of an order not exceeding m in k^2; the coefficient ρ_1 is however not zero, but

$$-nk^2\frac{(2r+1)(2r+2)}{1.2}-\frac{2r(2r+1)(2r+2)}{1.2.3}(1+k^2)$$
$$=-\frac{(2r+1)(2r+2)}{1.2}\left[nk^2+\frac{2r}{3}(1+k^2)\right].$$

We thus obtain for U_0 the expression

$$\nu U_0=\nu\psi_0(s)+n\delta\nu.\psi_1(s)+n^2\delta^2\nu.\psi_2(s)+\ldots.$$

But U_0 is a rational and integral function of order $n-1$. Hence we may omit in $\psi_2(s)$ all powers of s above s^{n-1}, and consequently all the functions $\delta^2\nu.\psi_2(s)$ after $\delta^{\frac{1}{2}(n-1)}\nu.\psi_{\frac{1}{2}(n-1)}(s)$; the higher powers of s disappearing of themselves. And if we denote by I the operating factor in the left-hand member of the equation (a) p. 40, increased by n^2k^2, viz.

$$I=(1-s^2)(1-k^2s^2)\frac{d^2}{ds^2}+[(2n-1)k^2-1-2(n-1)k^2s^2]\,s\frac{d}{ds}$$
$$+nk^2[1+(n-1)s^2]+4nk^2(1-k^2)\frac{d}{d.k^2},$$

we have the equations

$$I\psi_0(s)=0,\quad I\psi_1(s)=\psi_0(s),\quad I\psi_2(s)=\psi_1(s)\ldots$$

by which the functions $\psi_0(s)$, $\psi_1(s)$, ... may be successively calculated; viz. if

$$\psi_2(s)=C_r\frac{s^{2r}}{(2r)}+C_{r+1}\frac{x^{2r+2}}{(2r+2)\,!}\ldots,$$

we have

$$C_{\sigma+1}-[4\sigma^2+(4\sigma^2-4n\sigma-n)k^2]\,C_\sigma$$
$$+2\sigma(2\sigma-1)(2\sigma-n-1)(2\sigma-n-2)k^2C_{\sigma-1}+n\delta C_\sigma=B_\sigma,$$

if B_σ be the coefficient of $\frac{x^{2\sigma}}{2\sigma\,!}$ in $\psi_{r-1}(s)$.

IX.

On Modular Curves.

[These papers relate to the problem, &c., "Given a quadratic point, to find all the modular curves passing through it." &c., &c.

It ought to be worked into a memoir "On the ordinary multiple points of a modular curve; and on the intersections of two modular curves."]*

Let $[\alpha, \beta, \gamma]$ or ω be a given quadratic point, and let it be required to assign all the modular curves of any species passing through the point $X + iY = -\frac{1}{2} + \Phi(\omega)$.

Let $\begin{vmatrix} A, & B, & C \\ A', & B', & C' \end{vmatrix}$ be a pair of relatively prime solutions of the equation

$$\gamma A - 2\beta B + \alpha C = 0,$$

so that $$|\, \gamma, \ -2\beta, \ \alpha \,| = \begin{vmatrix} A, & B, & C \\ A', & B', & C' \end{vmatrix},$$

and so that all the solutions of this equation are comprised in the formula

$$\lambda' A - \lambda A', \quad \lambda' B - \lambda B', \quad \lambda' C - \lambda C'.$$

The determinants $B^2 - AC = D$, $B'^2 - A'C' = D'$ are positive, because $\beta^2 - \alpha\gamma = -\Delta$ is negative, and two pairs of conjugate imaginary points cannot be harmonic. Hence the symbols $[A, B, C]$, $[A', B', C']$ represent real semicircles passing through the point ω. The determinants of all semicircles passing through ω are included in the formula

$$D'\lambda^2 - 2J\lambda\lambda' + D\lambda'^2,$$

where $2J$ is the invariant $2BB' - AC' - A'C$, and where $J^2 - DD' = -\Delta$. The form $(D', -J, D)$ is the duplicate of the form (α, β, γ), and is transformed into the product of that form by itself by means of the bipartite linear substitution

$$\begin{vmatrix} A, & B, & B, & C \\ A', & B', & B', & C' \end{vmatrix}.$$

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* The words enclosed in [ ] are written on the back of the last page of the manuscript of the Note.

We thus obtain the theorem:

"The modular curves of order D pass through the point $[\alpha, \beta, \gamma]$ as often as there are primitive representations of D by the duplicate of the form (α, β, γ)."

To determine (by means of the corresponding semicircles) the ovals which pass through a given point $[\alpha, \beta, \gamma]$ we have the following rule:

"Let (P, Q, R) be the duplicate of (α, β, γ), and let

$$\begin{vmatrix} p_1, & q_1, & q_1, & r_1 \\ p_2, & q_2, & q_2, & r_2 \end{vmatrix}$$

be the substitution transforming (P, Q, R) into $(\alpha, \beta, \gamma)^2$; let also $(P, Q, R) \times (\mu_1, \mu_2)^2 = D$ be the given representation of D; the semicircle

$$[\mu_1 p_2 - \mu_2 p_1,\ \mu_1 q_2 - \mu_2 q_1,\ \mu_1 r_2 - \mu_2 r_1]$$

of determinant D passes through $[\alpha, \beta, \gamma]$."

To determine which of the modular curves of order D passes through $[\alpha, \beta, \gamma]$ we should have to distinguish the cases in which $\alpha, \beta, \gamma, \mu_1, \mu_2$ have different congruential values for the modulus 2. This discussion, for brevity, we omit.

If we apply to the form $(D', -J, D)$ the reducing substitution of Lagrange, we obtain the orders of the two lowest modular curves, which can pass through the point $-\frac{1}{2} + \Phi(\omega)$, or P. If on the tangents to these two curves at P we measure lengths PT, PT' equal to the square roots of the corresponding determinants, and if, completing the parallelogram $PTP'T'$ we construct the parallelism of which $PTP'T'$ is an elementary parallelogram, the tangents to the modular curves passing through P are the lines joining P to the nodes of this parallelism; and, if N be one of these nodes such that PN contains no other node besides P and N, PN is the square root of the determinant of the modular curve touching PN at P.

Gauss has shown that every class of the principal genus of any determinant is the duplicate of as many classes as there are ambiguous classes; i.e. classes producing the principal class by duplication. Thus no point of determinant $-\Delta$ can lie on a modular curve of order D unless D is represented by some class of the principal genus of determinant $-\Delta$; and, if there be such representations, there are, corresponding to each of them, as many points of determinant $-\Delta$ lying on the modular curves of order D, as there are sub-classes of determinant $-\Delta$ producing the principal class by duplication.

The necessary and sufficient condition that a given point $[\alpha, \beta, \gamma]$ should lie simultaneously on a modular curve of order D, and also on a modular curve of order D', is that D and D' should both be capable of primitive representation by the duplicate of $[\alpha, \beta, \gamma]$. Let

$$D = (P, Q, R) \times (\mu_1, \mu_2)^2, \quad D' = (P, Q, R) \times (\nu_1, \nu_2)^2$$

be the two representations; let $\begin{vmatrix} \mu_1, & \nu_1 \\ \mu_2, & \nu_2 \end{vmatrix} = v$, and let (P, Q, R) be transformed by $\begin{vmatrix} \mu_1, & \nu_1 \\ \mu_2, & \nu_2 \end{vmatrix}$ into $(D, -J, D')$ of determinant Δv^2. The form $(D, -J, D')$ is then transformed by $\begin{vmatrix} \nu_2, & -\nu_1 \\ -\mu_2, & \mu_1 \end{vmatrix}$ into $v^2 \times (P, Q, R)$; that is, it is transformed by the bipartite transformation

$$\begin{vmatrix} \nu_2, & -\nu_1 \\ -\mu_2, & \mu_1 \end{vmatrix} \times \begin{vmatrix} p_1, & q_1, & q_1, & r_1 \\ p_2, & q_2, & q_2, & r_2 \end{vmatrix}$$

into $v^2 \times (\alpha, \beta, \gamma)^2$. The semicircles

$$[\nu_2 p_1 - \nu_1 p_2, \ \nu_2 q_1 - \nu_1 q_2, \ \nu_2 r_1 - \nu_1 r_2],$$

$$[\mu_2 p_1 - \mu_1 p_2, \ \mu_2 q_1 - \mu_1 q_2, \ \mu_2 r_1 - \mu_1 r_2],$$

which are of determinants D', D, and which have J for their harmonic invariant and $v \times [\alpha, \beta, \gamma]$ for their covariant, are the two semicircles corresponding to the given representation and intersecting at the point $[\alpha, \beta, \gamma]$.

The equation $DD' = J^2 + v^2\Delta$ always subsists (as the preceding analysis implies) whenever any point of determinant Δ lies on two modular curves of the orders D and D'. But this condition, though necessary, is not sufficient; viz., confining ourselves to the case in which D and D' are relatively prime, J and v must be relatively prime, and the numbers D, D' must have each the characters of the principal genus of determinant $-\Delta$; when these additional conditions are satisfied, D and D' are necessarily represented by the same form of the principal genus; and the two sets of curves have as many points of determinant $-\Delta$ in common as there are ambiguous classes of determinant Δ.

Let $[\alpha_1, \beta_1, \gamma_1]$ and $[\alpha_2, \beta_2, \gamma_2]$ be two quadratic points ω_1, ω_2, of which the determinants are to one another as two squares; let $\Delta_1 = \Delta'\theta_1^2$, $\Delta_2 = \Delta'\theta_2^2$, Δ' being the greatest

common divisor of Δ_1 and Δ_2; and let it be required to assign all the modular curves with regard to which $-\frac{1}{2}+\Phi(\omega_1)$ and $-\frac{1}{2}+\Phi(\omega_2)$ are inverse. Let (P, Q, R) be a form of determinant Δ' compounded of $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$. If $-\frac{1}{2}+\Phi(\omega_1)$ and $-\frac{1}{2}+\Phi(\omega_2)$ are inverse with regard to any modular curve of determinant D, D is divisible by $\theta_1\theta_2$, and the quotient is capable of primitive representation by (P, Q, R). Conversely, if these conditions are satisfied, a modular curve of determinant D exists with regard to which the two points are inverse. Let $(P, Q, R)\times(\mu_1, \mu_2)^2=\dfrac{D}{\theta_1\theta_2}$; and let

$$\begin{vmatrix} p_1, & q_1, & q'_1, & r_1 \\ p_2, & q_2, & q'_2, & r_2 \end{vmatrix}$$

be the substitution transforming (P, Q, R) into

$$(\alpha_1, \beta_1, \gamma_1)\times(\alpha_2, \beta_2, \gamma_2);$$

the points $\omega_1=x_1+iy_1$, $\omega_2=x_2+iy_2$ will satisfy the equation

$$-x_2+iy_2=\begin{vmatrix} \mu_1 q'_2-\mu_2 q'_1, & \mu_1 r_2-\mu_2 r_1 \\ \mu_1 p_2-\mu_2 p_1, & \mu_1 q_2-\mu_2 q_1 \end{vmatrix}\times(x_1+iy_1),$$

the determinant of the matrix being D; i.e. $-\frac{1}{2}+\Phi(\omega_1)$ and $-\frac{1}{2}+\Phi(\omega_2)$ are inverse with regard to a modular curve of determinant D. If D and D' are two uneven numbers relatively prime, the necessary and sufficient condition that two given points should be inverse with regard to modular curves appertaining to each of those determinants is that the two points should have the same determinant Δ, and that D and D' should be primitively represented by the same class of determinant Δ; or, which is the same thing, that $D\times D'$ should be represented by the principal class of determinant Δ. If this condition be satisfied, every point of determinant Δ is inverse to another point of that determinant with regard to a modular curve (D) and also with regard to a modular curve (D'). If the class by which D and D' are represented is not a class of the principal genus, all the points of intersection thus obtained are imaginary, no two inverse points coinciding. But if D and D' are represented by a class Γ of the principal genus, the classes which by their duplication produce the class Γ give real points of intersection.

X.

On the Quarter Periods K, $\frac{1}{2}iK'$.

The fundamental pair of quarter periods (K, $\frac{1}{2}iK'$) is not in general the absolutely least pair of quarter periods appertaining to the doubly periodic function; for it does not follow that the parallelism (K, $\frac{1}{2}iK'$) is absolutely reduced because the parallelism (K, iK') is primarily reduced.

Problem. **To determine, for any given value of k^2, the absolutely reduced parallelism equivalent to (K, iK').**

From the preceding discussion it appears that the reduced triangle is formed by the vectors

$$K, \quad iK', \quad -K-iK',$$

or

$$K-iK', \quad iK', \quad -K,$$

according as the angle from K to iK' is obtuse or acute. It may be proved by the considerations already employed in the demonstration of the theorem(...)*, or by a method presently to be explained, that this angle is obtuse or acute according as the coefficient of i in k^2 is negative or positive.

We have now to show how, for any given value of k^2, the vectors of the reduced triangle can be arranged in order of magnitude. For this purpose we employ the principles contained in a Memoir "Sur les équations modulaires," which will be found in the Transactions of the *Accademia dei Lincei*, vol. I. Ser. III. (1877); it will suffice to consider only one of the propositions to be demonstrated. We understand by $[z]$ the absolute value, i.e. the analytical modulus, of any complex quantity z.

"The inequalities

$$[K] < [K \pm iK']$$

subsist simultaneously, if $[k^2] < 1$; if $[k^2] > 1$, we have

$$[K-iK'] < [K] < [K+iK'],$$

or

$$[K+iK'] < [K] < [K-iK'],$$

according as the coefficient of i in the imaginary part of k^2 is positive or negative."

Let $\dfrac{iK'}{K} = X + iY$; the inequalities $[K] \lesseqgtr [K + iK']$ are equivalent to the inequalities $0 \lesseqgtr 2X + X^2 + Y^2$ But the

* Blanks in the manuscript are denoted by dots enclosed in parentheses; thus (...).

quadrant of the circle $X^2+Y^2+2X=0$, which lies within the reduced space defined by the inequalities in question, and which runs from the cusp at 0 to the point $-1+i$, is represented in the plane $k^2=\frac{1}{2}+x+iy$ by a semicircle of radius 1, running below the axis of x from the point $x=+\frac{1}{2}$, to the point $x=-\frac{3}{2}$; and the two regions containing the points -1 and $+1$ respectively, into which the reduced space is divided by the quadrant are represented in the plane of xy by the two regions, containing the points infinitely far off on the axis of y in the negative and positive directions respectively, into which that plane is divided by the axis of x from $+\infty$ to $\frac{1}{2}$, by the semicircle, and by the axis of x from $-\frac{3}{2}$ to $-\infty$. Hence according as $k^2-\frac{1}{2}$ lies in the first or second of these regions or on the boundary between them, we have

$$[K]>[K+iK'], \text{ or } [K]<[K+iK'], \text{ or } [K]=[K+iK'].$$

Similarly, if we divide the plane xy into two regions, one lying above the axis of x and external to the circle

$$(x+\tfrac{1}{2})^2+y^2=1,$$

the other comprising the rest of the plane, it will be found that according as $k^2-\frac{1}{2}$ lies in the first or second of these regions, or on the boundary between them, we have

$$[K]>[K-iK'], \text{ or } [K]<[K-iK'], \text{ or } [K]=[K-iK'].$$

These two results taken together are equivalent to the proposition which we have enunciated.

To complete the solution of the problem we have to discuss the inequalities $[iK']\gtrless[K\pm iK']$; their theory depends on the representation of the lines $2X\pm 1=0$, or, rather, of those portions of them which lie within the reduced space, by the semicircles $(x-\frac{1}{2})^2+y^2=1$.

The final result is perhaps most simply expressed in the following form: The plane xy is divided by the axes and by the circles $(x\pm\frac{1}{2})^2+y^2=1$ into twelve regions. The regions within both circles are designated by A, those within one circle only by B, those outside both by C; the four regions A are numbered 1, 2, 3, 4 according to the quadrants in which they lie; the regions B and C are similarly distinguished (see fig. 1). The Table indicates the arrangement of the four periods K, iK', $K+iK'$, $K-iK'$, in ascending order of absolute magnitude, according to the region in which the vector point $x+iy=k^2-\frac{1}{2}$ is situated.

A.

(1) iK', K, $K-iK'$, $K+iK'$;
(2) K, iK', $K-iK'$, $K+iK'$;
(3) K, iK', $K+iK'$, $K-iK'$;
(4) iK', K, $K+iK'$, $K-iK'$.

B.

(1) iK', $K-iK'$, K, $K+iK'$;
(2) K, $K-iK'$, iK', $K+iK'$;
(3) K, $K+iK'$, iK', $K-iK'$;
(4) iK', $K+iK'$, K, $K-iK'$.

C.

(1) $K-iK'$, iK', K, $K+iK'$;
(2) $K-iK'$, K, iK', $K+iK'$;
(3) $K+iK'$, K, iK', $K-iK'$;
(4) $K+iK'$, iK', K, $K-iK'$.

It will be seen that along the axis of x,

$$[K-iK']=[K+iK'];$$

along the axis of y, $\quad [iK']=[K]$;

along the upper and lower semicircles of $(x+\frac{1}{2})^2+y^2=1$,

$$[K]=[K-iK'],\ [K]=[K+iK'] \text{ respectively};$$

along the upper and lower semicircles of $(x-\frac{1}{2})^2+y^2=1$,

$$[iK']=[K-iK'],\ [iK']=[K+iK'] \text{ respectively};$$

and that when we traverse any one of these six boundaries the quantities, which on it are equal, change places with one another.

Cor. 1. The zeros of the Theta functions

$$\vartheta_1\left(\frac{\pi x}{2K}\right),\quad \vartheta_2\left(\frac{\pi x}{2K}\right),\quad \vartheta_3\left(\frac{\pi x}{2K}\right),\quad \vartheta_0\left(\frac{\pi x}{2K}\right),$$

being respectively

$$2rK+2siK',\quad (2r+1)K+2siK',$$
$$(2r+1)K+(2s+1)iK',\quad 2rK+(2s+1)iK',$$

it follows that the least zeros of $\vartheta_2\left(\frac{\pi x}{2K}\right)$ are always $\pm K$, and the least zeros of $\vartheta_0\left(\frac{\pi x}{2K}\right)$ are always $\pm iK'$. But the least zeros of $\vartheta_3\left(\frac{\pi x}{2K}\right)$ are $\pm(K+iK')$ or $\pm(K-iK')$, according as the coefficient of i in the imaginary part of k^2 is negative or positive; and the least zeros of $\vartheta_1\left(\frac{\pi x}{2K}\right)$ are $\pm 2K$, $\pm 2iK'$, $\pm 2(K-iK')$, $\pm 2(K+iK')$, according as $k^2-\frac{1}{2}$ lies (1) inside the circle $(x+\frac{1}{2})^2+y^2=1$ to the left of the axis of y, (2) inside the circle $(x-\frac{1}{2})^2+y^2=1$ to the right of the axis of y, (3) outside the two circles and above the axis of x, (4) outside the two circles and below the axis of x.

These determinations assign in all cases the circles of convergence of the developments of

$$\frac{\sin\operatorname{am}u}{u}, \quad \cos\operatorname{am}u, \quad \Delta\operatorname{am}u,$$

and their reciprocals, in series proceeding by powers of u.

Cor. 2. The Table also assigns in every case the absolutely least pair of conjugate periods of the functions $\sin^2\operatorname{am}u$, $\cos^2\operatorname{am}u$, $\Delta^2\operatorname{am}u$; viz., this pair consists of the least and least but one of the four quantities $2K$, $2iK'$, $2(K\pm iK')$. But to obtain the absolutely least pairs of periods appertaining to the functions $\sin\operatorname{am}u$, $\cos\operatorname{am}u$, and $\Delta\operatorname{am}u$ themselves, we have to consider the modular curves of the square determinant 4, instead of the lines and circles of determinant $+1$.

Problem. To determine, for any given value of k^2, the absolutely reduced parallelisms equivalent to $(K, \frac{1}{2}iK')$, $(\frac{1}{2}[K-iK'], \frac{1}{2}[K+iK'])$, and $(\frac{1}{2}K, iK')$.

Since the reduced triangle of the parallelism (K, iK') is acute-angled, one of the two triangles into which it is divided by the line joining any of its vertices to the middle point of the opposite side is certainly acute-angled, and the half side is always less than the bisecting line. Hence the reduced triangle of the parallelism $(K, \frac{1}{2}K')$ is one of the four

$$K, \quad \tfrac{1}{2}iK', \quad -K-\tfrac{1}{2}iK', \qquad (1)$$

$$K+\tfrac{1}{2}iK', \quad \tfrac{1}{2}iK', \quad -K-iK', \qquad (2)$$

$$K-\tfrac{1}{2}iK', \quad \tfrac{1}{2}iK', \quad -K, \qquad (3)$$

$$K-iK', \quad \tfrac{1}{2}iK', \quad -K+\tfrac{1}{2}iK', \qquad (4)$$

in each of which the side $\frac{1}{2}iK'$ is less than the side $K \pm \frac{1}{2}iK'$. In fact, we have the case (1), (2), (3), or (4), according as

(1) amplitude of $\frac{iK'}{K} > \frac{1}{2}\pi$; $[K] < [K + iK']$;

(2) amplitude of $\frac{iK'}{K} > \frac{1}{2}\pi$; $[K] > [K + iK']$;

(3) amplitude of $\frac{iK'}{K} < \frac{1}{2}\pi$; $[K] < [K - iK']$;

(4) amplitude of $\frac{iK'}{K} < \frac{1}{2}\pi$; $[K] > [K - iK']$.

These inequalities have been already examined; in addition to these we have to consider the inequalities

$$[K] \lesseqgtr [\tfrac{1}{2}iK']; \qquad [K] \lesseqgtr [K \pm iK'];$$
$$[K \pm iK] \lesseqgtr [K \pm iK']; \quad [K \pm iK] \lesseqgtr [\tfrac{1}{2}iK'];$$

which determine the order of magnitude of the sides in the four triangles; and which are equivalent to the following:

$$X^2 + Y^2 \gtrless 4; \qquad X^2 + Y^2 \pm 4X \gtrless 0.$$
$$3(X^2 + Y^2) \pm 4X \lesseqgtr 0; \quad 3(X^2 + Y^2) \pm X + 4 \lesseqgtr 0;$$

In the plane $k^2 = \frac{1}{2} + x + iy$, the circle $X^2 + Y^2 = 4$, and the three pairs of circles

$$X^2 + Y^2 \pm 4X = 0, \quad 3(X^2 + Y^2) \pm 4X = 0, \quad 3(X^2 + Y^2) \pm 8X + 4 = 0,$$

are represented by the four loops of the modular curve (...), beginning with the innermost and proceeding in order to the outermost. If we designate the regions (taken in the same order from within outwards) into which the plane is divided by the curve by A, B, C, D, E, the Table gives the three least quarter periods corresponding to a value of $k^2 - \frac{1}{2}$ lying within any given region:

A.

$K, \ \frac{1}{2}iK'; \ \ K \pm \frac{1}{2}iK'$.

B.

$-\frac{1}{2}iK', \ K; \ \ K \pm \frac{1}{2}iK'$.

C_1.

$-\frac{1}{2}iK', \ K \pm \frac{1}{2}iK'; \ \ K$.

C_2.

$-\frac{1}{2}iK', \ K \pm \frac{1}{2}iK'; \ \ K \pm iK'$.

D.

$-\frac{1}{2}iK', \ K \pm iK'; \ \ K \pm \frac{1}{2}iK'$.

E.

$K \pm iK', \ \frac{1}{2}iK'; \ \ K \pm \frac{1}{2}iK'$.

The upper signs are to be taken in the region below the axis of x, and *vice versâ*. The region C is supposed to be divided into two, C_1 and C_2, the first within, the second without the circle $(x+\frac{1}{2})^2+y^2=1$.

To obtain the reduced parallelism equivalent to $(\frac{1}{2}K, iK')$ we have only to take the modular curve (...) which is symmetrical to (...) with respect to the axis of y, and to interchange K and K', dividing at the same time by i. We thus obtain the following Table:

A.

$$-iK',\ \tfrac{1}{2}K;\quad iK'\pm\tfrac{1}{2}K.$$

B.

$$\tfrac{1}{2}K,\ iK';\quad iK'\pm\tfrac{1}{2}K.$$

C_1.

$$\tfrac{1}{2}K,\ iK'\pm\tfrac{1}{2}K;\quad iK'.$$

C_2.

$$\tfrac{1}{2}K,\ iK'\pm\tfrac{1}{2}K;\quad iK'\pm K.$$

D.

$$\tfrac{1}{2}K,\ iK'\pm K;\quad iK'\pm\tfrac{1}{2}K.$$

E.

$$iK'\pm K,\ \tfrac{1}{2}K;\quad iK'\pm\tfrac{1}{2}K.$$

The reduction of the parallelism $(\frac{1}{2}[K-iK'], \frac{1}{2}[K+iK'])$ depends on the remaining modular curve of determinant 4, ... which is symmetrical with regard to both axes. The reduced triangle of this parallelism always has the vectors $\frac{1}{2}(K\pm iK')$ for two of its sides, and either K or iK' for its third side: in this enunciation, which is obtained by bisecting the side $K\pm iK'$ of the reduced triangle of the parallelism (K, iK'), the signs of the vectors are neglected. Therefore, besides the inequalities $[K]\gtrless[iK']$, $[K+iK']\gtrless[K-iK']$, which we have already considered, we have to examine the inequalities

$$[K]\gtrless\tfrac{1}{2}[K\pm iK'];\quad [iK']\gtrless\tfrac{1}{2}[K\pm iK'];$$

or, which is the same thing, the inequalities

$$(X\pm 1)^2+Y^2\lesseqgtr 4;\quad (X\pm 1)^2+Y^2\lesseqgtr 4(X^2+Y^2).$$

The four circles

$$X^2+Y^2\pm 2X-3=0;\quad 3(X^2+Y^2)\pm 2X-1=0;$$

or rather the arcs of those circles which lie within the reduced space are represented in the plane $k^2-\frac{1}{2}=x+iy$ by the curve (...), of which the form may be roughly compared to that of a hyperbola having an internal loop at each vertex.

We designate by A the central infinite region between the two branches of the curve; by B and B' the infinite regions internal to the two branches of the curve on the right and left of the axis of y respectively; by C and C' the regions internal to the loops: we then have the following scheme, the upper signs being taken in the upper part of the plane, and *vice versâ*:

$$A.$$
$$\tfrac{1}{2}(K \pm iK'),\ \tfrac{1}{2}(\mp K + iK');\quad K,\ iK.$$
$$A'.$$
$$\tfrac{1}{2}(K \pm iK'),\ \tfrac{1}{2}(\mp K + iK');\quad iK',\ K.$$
$$B.$$
$$\tfrac{1}{2}(K \pm iK'),\ \mp K;\quad \tfrac{1}{2}(K \mp iK'),\ iK.$$
$$B'.$$
$$\tfrac{1}{2}(K \pm iK'),\ iK';\quad \tfrac{1}{2}(K \mp iK'),\ K.$$
$$C.$$
$$K,\ \tfrac{1}{2}(\pm K + iK');\quad \tfrac{1}{2}(K \mp iK'),\ iK'.$$
$$C'.$$
$$-iK',\ \tfrac{1}{2}(K \pm iK');\quad \tfrac{1}{2}(K \mp iK'),\ K.$$

ON THE LINEAR TRANSFORMATION OF THE THETA FUNCTIONS.

By Professor *Cayley*.

THE functions referred to are the single theta Functions; these may be defined as doubly infinite products, as was in fact done in my "Memoire sur les fonctions doublement periodiques," Liouv. t. x. (1845) pp. 385–420; and it is interesting to consider from this point of view the theory of their linear transformation: this I propose to do in the present paper, adopting throughout the notation of Smith's "Memoir on the Theta and Omega Functions."

The periods K, iK' are in general imaginary quantities

$$K = A + Bi,$$
$$iK' = C + Di,$$

where $AD-BC$ is positive; writing then $\omega=\frac{iK'}{K}$, and $q=e^{i\pi\omega}$, also for shortness

$$(q1)=2q^{\frac{1}{4}}\Pi_1^{\infty}(1-q^{2n})^3.$$

where $q^{\frac{1}{4}}$ denotes $e^{\frac{1}{4}(i\pi\omega)}$, the expression of the odd theta-function $\vartheta_1(x, \omega)$ as a doubly infinite product is

$$\vartheta_1(x, \omega)=(q1)\,x\,\Pi\Pi\left(1+\frac{x}{m\pi+n\omega\pi}\right), \quad \left(\frac{\mu}{\nu}=\infty\right),$$

where (m, n) have any positive or negative integer values (the combination $m=0$, $n=0$ excluded) from $m=-\mu$ to μ, and $n=-\nu$ to ν, μ and ν being each ultimately infinite but so that μ is infinite in comparison with ν; this condition in regard to the limits is indicated by $\mu/\nu=\infty$; and similarly $\nu/\mu=\infty$ would indicate that ν was infinite in comparison with μ.

The condition as to the limits might be that (m, n) have any positive or negative values (excluding as before) such that the modulus of $m+n\omega$ does not exceed a positive value T, which is ultimately taken to be infinite; this condition may be indicated by mod $=\infty$.

The values of the double product corresponding to the different conditions as to the limits are not equal, but they differ only by an exponential factor, the exponent being a multiple of x^2; we thus have

$$x\Pi\Pi\left(1+\frac{x}{mK+niK'}\right)\left(\frac{\mu}{\nu}=\infty\right)$$
$$=\exp(\nabla x^2).x\Pi\Pi\left(1+\frac{x}{mK+niK'}\right)(\text{mod}=\infty)$$

where ∇ is a determinate value, depending on K and K'; and similarly

$$x\Pi\Pi\left(1+\frac{x}{m\Lambda+ni\Lambda'}\right)\left(\frac{\mu}{\nu}=\infty\right)$$
$$=\exp(\square x^2)\,x\Pi\Pi\left(1+\frac{x}{m\Lambda+ni\Lambda'}\right), (\text{mod}=\infty),$$

where $\square$ is a determinate value depending in like manner on Λ, Λ'.

We have, then, as above

$$\vartheta_1(x, \omega) = (q1)\,x\Pi\Pi\left(1+\frac{x}{m\pi+n\omega\pi}\right)\left(\frac{\mu}{\nu}=\infty\right)$$

$$= (q1)\frac{\pi}{K}\frac{Kx}{\pi}\Pi\Pi\left(1+\frac{\frac{Kx}{\pi}}{mK+niK'}\right)$$

$$= (q1)\frac{\pi}{K}\exp\left(\nabla\frac{k^2x^2}{\pi^2}\right)\frac{Kx}{\pi}\Pi\Pi\left(1+\frac{\frac{Kx}{\pi}}{mK+niK'}\right),$$

$$(\text{mod} = \infty),$$

viz. we have thus defined $\vartheta_1(x, \omega)$ as a doubly infinite product with the limiting condition (mod $=\infty$); if for x we write $\frac{\pi x}{h}$, h arbitrary, we have

$$\vartheta_1\left(\frac{\pi x}{h}, \omega\right) = (q1)\frac{\pi}{K}\exp\left(\nabla\frac{K^2x^2}{h^2}\right)\frac{Kx}{h}\Pi\Pi\left(1+\frac{\frac{Kx}{h}}{mK+niK'}\right),$$

$$(\text{mod} = \infty),$$

and similarly if $\Omega = \frac{i\Lambda'}{\Lambda}$, $Q = e^{i\pi\Omega}$, then

$$\vartheta_1\left\{(a+b\Omega)\frac{\pi x}{h}, \Omega\right\} = (Q1)\frac{\pi}{\Lambda}\exp\left\{(a+b\Omega)^2 \,\square\, \frac{\Lambda^2x^2}{h^2}\right\}$$

$$\times (a+b\Omega)\frac{\Lambda x}{h}\Pi\Pi\left(1+\frac{(a+b\Omega)\frac{\Lambda x}{h}}{m\Lambda+ni\Lambda'}\right)(\text{mod} = \infty).$$

In the case of a linear transformation we have

$$\omega = \begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \times \Omega, \text{ that is } \omega = \frac{c+d\Omega}{a+b\Omega},$$

where a, b, c, d are positive or negative integers such that $ad-bc=+1$; it is to be shown that the two infinite products are in this case identical; this being so, we have

$$\frac{\vartheta_1\left\{(a+b\Omega)\frac{\pi x}{h}, \Omega\right\}}{\vartheta\left\{\frac{\pi x}{h}, \omega\right\}} = \frac{(Q1)}{(q1)}\exp\left\{\{(a+b\Omega)^2 \,\square\, \Lambda^2 - \nabla K^2\}\frac{x^2}{h^2}\right\},$$

viz. the two functions differ only by a constant factor and by an exponential factor, the exponent being a multiple of x^2; after all reductions this factor is found to be

$$= \exp\left(-i\pi b\,(a+b\Omega)\,\frac{x^2}{h^2}\right).$$

We have
$$\omega = \frac{c+d\Omega}{a+b\Omega},$$

or since
$$\omega = \frac{iK'}{K},\quad \Omega = \frac{i\Lambda'}{\Lambda},$$

this is
$$\frac{iK'}{K} = \frac{c\Lambda + di\Lambda'}{a\Lambda + bi\Lambda'},$$

or say
$$\frac{1}{M}\,K = a\Lambda + bi\Lambda',$$

$$\frac{1}{M}iK' = c\Lambda + di\Lambda',$$

(either of which equation may be taken as a definition of the multiplier M). We have

$$\frac{K}{M\Lambda} = a + b\Omega,$$

$$\frac{1}{M}(mK' + niK') = (am + cn)\,\Lambda + (bm + dn)\,i\Lambda'$$
$$= m'\Lambda + n'i\Lambda',$$

if
$$m' = am + cn,$$
$$n' = bm + dn.$$

Here to any integer values of (m, n) there correspond integer values of m', n'; and conversely, in virtue of the equation $ad - bc = 1$, to any integer values of m', n' there correspond integer values of m, n. The two products are

$$\Pi\Pi\left(1 + \frac{\dfrac{Kx}{Mh}}{m'\Lambda + n'i\Lambda'}\right),\ (\text{mod} = \infty),$$

$$\Pi\Pi\left(1 + \frac{\dfrac{(a+b\Omega)\,\Lambda x}{h}}{m\Lambda + ni\Lambda'}\right),\ (\text{mod} = \infty),$$

But, as above, we have $\frac{K}{M}=(a+b\Omega)\Lambda$, and then observing that in the first of the two products we may for m', n' write m, n; it at once appears that the two products are identical.

The exponential factor, writing therein $(a+b\Omega)\Lambda=\frac{K}{M}$, becomes

$$\exp\left\{\left(\frac{\square}{M^2}-\nabla\right)\frac{K^2x^2}{h^2}\right\}.$$

The values of ∇, $\square$ are at once obtained by means of a formula given in the Memoir in Liouville, p. 396, viz. we have

$$\nabla=-\tfrac{1}{2}(B+\beta),$$

where
$$B=\frac{\pi(\omega\upsilon+\omega'\upsilon')}{\Omega\Upsilon\bmod(\omega\upsilon'-\omega'\upsilon)},$$

(see p. 395, the denominator-factor $\Omega\Upsilon$ is by mistake omitted, p. 396),

$$\beta=\frac{\pi i}{\Omega\Upsilon}\frac{(\omega\upsilon'-\omega'\upsilon)}{\bmod(\omega\upsilon'-\omega'\upsilon)},$$

and comparing with the present notation

$$\Omega=\omega+\omega' i,\quad =A+Bi=K,$$

$$\Upsilon=\upsilon+\upsilon' i,\quad =C+Di=K'i,$$

so that Ω, Υ denote K, $K'i$, and ω, ω', υ, υ' denote A, B, C, D respectively: $\omega\upsilon'-\omega'\upsilon$ is thus $=AD-BC$, which has been assumed to be positive; hence also $\bmod(\omega\upsilon'-\omega'\upsilon)=AD-BC$, and the formula becomes

$$\nabla=-\tfrac{1}{2}\pi\left\{\frac{AC+BD}{i(AD-BC)}+1\right\}\frac{1}{KK'}.$$

Now writing

$$\Lambda=A_1+B_1 i,$$

$$i\Lambda'=C_1+D_1 i,$$

then we have

$$\frac{1}{M}(A+Bi)=a\Lambda+bi\Lambda'=a(A_1+B_1 i)+b(C_1+D_1 i),$$

$$\frac{1}{M}(C+Di)=c\Lambda+di\Lambda'=c(A_1+B_1 i)+d(C_1+D_1 i);$$

consequently if $\quad M = \rho(\cos\theta + i\sin\theta)$

we have

$$\frac{1}{\rho}(\ \ A\cos\theta + B\sin\theta) = aA_1 + bC_1,$$

$$\frac{1}{\rho}(-A\sin\theta + B\cos\theta) = aB_1 + bD_1,$$

$$\frac{1}{\rho}(\ \ C\cos\theta + D\sin\theta) = cA_1 + dC_1,$$

$$\frac{1}{\rho}(-C\sin\theta + D\cos\theta) = cB_1 + dD_1,$$

and thence

$$\frac{1}{\rho^2}(AD - BC) = (ad - bc)(A_1D_1 - B_1C_1), \quad = (A_1D_1 - B_1C_1).$$

Hence $A_1D_1 - B_1C_1$ is positive, and we have

$$\square = -\tfrac{1}{2}\pi\left\{\frac{A_1C_1 + B_1D_1}{i(A_1D_1 - B_1C_1)} + 1\right\}\frac{1}{\Lambda\Lambda'}.$$

Take K_1 the conjugate of K, Λ_1 the conjugate of Λ, then

$$K_1 = A - Bi, \quad \Lambda_1 = A_1 - B_1 i,$$

$$iK' = C + Di, \quad i\Lambda' = C_1 + D_1 i.$$

We have

$$iK_1K' = AC + BD + i(AD - BC),$$

and therefore

$$\frac{K_1K'}{(AD - BC)} = \frac{AC + BD}{i(AD - BC)} + 1, \quad \nabla = \frac{-\tfrac{1}{2}\pi}{AD - BC}\frac{K_1}{K},$$

and similarly

$$\square = \frac{-\tfrac{1}{2}\pi}{A_1D_1 - B_1C_1}\frac{\Lambda_1}{\Lambda}.$$

The exponential is $\left(\frac{\square}{M^2} - \nabla\right)\frac{K^2x^2}{h^2}$; and we have

$$\frac{\square}{M^2} - \nabla = \frac{-\tfrac{1}{2}\pi}{M^2(A_1D_1 - B_1C_1)}\frac{\Lambda_1}{\Lambda} + \frac{\tfrac{1}{2}\pi}{AD - BC}\frac{K_1}{K},$$

which is

$$= \frac{-\frac{1}{2}\pi}{M^2(A_1D_1-B_1C_1)}\frac{\Lambda_1}{\Lambda} + \frac{\frac{1}{2}\pi}{\rho^2(A_1D_1-B_1C_1)}\frac{K_1}{K},$$

$$= \frac{-\frac{1}{2}\pi}{A_1D_1-B_1C_1}\left(\frac{1}{M^2}\frac{\Lambda_1}{\Lambda} - \frac{1}{\rho^2}\frac{K_1}{K}\right).$$

But $\rho/M = \cos\theta - i\sin\theta$, or calling this for a moment P, then $1/M^2 = P^2/\rho^2$, and the formula may be written

$$\frac{\Box}{M^2} - \nabla = \frac{-\frac{1}{2}\pi P}{\rho^2(A_1D_1-B_1C_1)}\left(P\frac{\Lambda_1}{\Lambda} - P^{-1}\frac{K_1}{K}\right)$$

$$= \frac{-\frac{1}{2}\pi P}{\rho^2(A_1D_1-B_1C_1)}\{(\cos\theta - i\sin\theta)\,\Lambda_1 K$$

$$- (\cos\theta + i\sin\theta)\,\Lambda K_1\}\frac{1}{K\Lambda}.$$

The term in { } is

$$(\cos\theta - i\sin\theta)(A+Bi)(A_1-B_1i) - (\cos\theta + i\sin\theta)(A-Bi)(A_1+B_1i),$$

$$= 2\cos\theta[-(AB_1 - A_1B)\,i] - 2i\sin\theta\,(AA_1 + BB_1),$$

$$= -2i\{(AB_1 - A_1B)\cos\theta + (AA_1 + BB_1)\sin\theta\},$$

$$= -2i\{B_1(A\cos\theta + B\sin\theta) - A_1(-A\sin\theta + B\cos\theta)\},$$

$$= -2i\rho\{B_1(aA_1 + bC_1) - A_1(aB_1 + bD_1)\},$$

$$= +2i\rho b\,(A_1D_1 - B_1C_1).$$

Hence

$$\frac{\Box}{M^2} - \nabla = \frac{-\frac{1}{2}\pi P}{\rho^2(A_1D_1-B_1C_1)}\,2ib\rho\,(A_1D_1-B_1C_1)\frac{1}{K\Lambda}$$

$$= -\frac{i\pi bP}{\rho}\frac{1}{K\Lambda} = -\frac{i\pi b}{M}\frac{1}{K\Lambda},$$

and the exponential thus is

$$= \exp\left(-\frac{i\pi b}{M}\frac{1}{K\Lambda}\frac{K^2x^2}{h^2}\right), \quad = \exp\left(-i\pi b\frac{K}{M\Lambda}\frac{x^2}{h^2}\right);$$

or, since $\frac{K}{M\Lambda} = (a + b\Omega)$, this is

$$= \exp\left(-i\pi b\,(a+b\Omega)\frac{x^2}{h^2}\right);$$

and we have thus the required formula

$$\frac{\vartheta_1\left\{(a+b\Omega)\frac{\pi x}{h},\ \Omega\right\}}{\vartheta_1\left\{\frac{\pi x}{h},\ \Omega\right\}} = \frac{(Q1)}{(q1)}(a+b\Omega)\exp\left(-i\pi b\,(a+b\Omega)\frac{x^2}{h^2}\right).$$

ON A PORISM RELATING TO CIRCLES.

By *J. Larmor.*

1. PONCELET'S porism of the in- and circum-scribed polygon is most easily proved thus:—Let the sides AB, BC, ... of the polygon $ABC...KA$ inscribed in the one circle touch the other circle in $\alpha, \beta, \gamma, ... \kappa$ respectively; then if we draw another such polygon $A'B'C''...K'A''$, starting at A', a point very near to A, we see at once that

$$AA' : BB' = A\alpha : B\alpha, \quad BB' : CC' = B\beta : C\beta,$$

and similar relations, until finally we have

$$KK' : AA'' = K\kappa : A\kappa;$$

so that, multiplying, we have $AA'' = AA'$ (since the tangents to the second circle are equal in pairs), and therefore A', A'' coincide, which proves that the second polygon is closed.

It is to be noticed that the two properties on which the proof depends still remain true when the sides of the polygon are *arcs of equal circles* instead of straight lines. In this case the second intersections of these equal circles plainly lie on another circle, which is the inverse of the circumscribed with respect to the centre of the inscribed circle of the polygon. Further, the equal circles may be specified as circles which touch the inscribed circle and another concentric with it, or as circles which cut each of *two* circles concentric with it at equal angles. Generalizing the result now by the method of inversion, we have the theorem that—If a system of circles be drawn, each touching two given circles (or cutting them at given angles), and each intersecting the consecutive one in a point which lies on another given circle, the other intersections of consecutive ones will lie on a fourth fixed circle; and if one system of such circles be re-entrant, every system will be re-entrant.

2. This seems to include most of the ordinary porisms relating to circles. In particular:

(1) If one of the circles touched be the circle at infinity, we have Poncelet's original theorem;

(2) If three pairs of the system of circles touch, the two circles on which the intersections lie meet in three points, and therefore coincide, and we have the theorem that if one ring

of circles can be drawn each touching two fixed circles, and its two neighbours in the ring, an infinite number of such rings can be drawn;

(3) If one of the circles touched becomes a point, we have the inverse of (1);

(4) One of the circles touched may become a straight line;

(5) The circles may touch one circle and cut another at right angles: this latter may become a line, in which case the circles touch a fixed circle and have their centres on a fixed straight line; and so on.

In each case the number of circles in the variable system is unalterable, and may be any from 3 upwards.

3. It is clear, from exactly similar considerations, that the theorem is true for circles drawn on the surface of a sphere. Indeed this may be inferred at once by inverting the plane of the figure into a sphere. When the two small circles touched by the system are equal and parallel, we have the analogue of Poncelet's theorem.

4. In the plane theorem the method of projection allows us to substitute conics passing through two fixed points for circles, and the method of reciprocation allows us to substitute common tangents for common points, in the enunciation.

April 16, 1883.

MATHEMATICAL NOTES.

By *Arthur Buchheim*, *B.A.*, late Scholar of New College, Oxford.

1. *Orthogonal transformations.*

Using Prof. Cayley's notation of matrices, and denoting by (A) the matrix conjugate to A, the matrix A is orthogonal, *i.e.* defines an orthogonal substitution, if $A(A)=1$, where 1 denotes the matrix unity (the identical substitution). Cayley's solution of the problem of orthogonal transformations is a particular case of the following: Given $C=AB$, to find under what conditions C is orthogonal. We have $(C)=(B)(A)$; and therefore, if C is orthogonal, we must have $AB(B)(A)=1$; this gives $B(B)=A^{-1}(A^{-1})$. In Cayley's solution we take $x=B\xi$, $X=(B)\xi$; so that $x=B(B^{-1})X$ and the matrix $B(B^{-1})$ is orthogonal if $(B^{-1})B^{-1}=B^{-1}(B^{-1})$;

this gives $(B)B = B(B)$; and if, as in Cayley's solution, we have $B + (B) = 2$, this condition is obviously satisfied.

2. *Poisson's transformation of the equations of dynamics.*

This transformation is the well-known one from $p' = \frac{dq}{dt}$ to $p = \frac{\partial T}{\partial q}$, T denoting the kinetic energy. The object of this note is the derivation of the equations

$$q' = \frac{\partial T'}{\partial p}; \quad \frac{\partial T'}{\partial q} = -\frac{\partial T}{\partial q} \{T' = T(q\ p)\},$$

immediately from the expression of T as a quadric function of the q's.

We start with a function

$$T = T(x_1 \ldots x, y_1 \ldots y_n) = a^2,$$

so that the coefficients a_{ik} are functions of the x's. Let Δ denote the discriminant of the quadric; let $a_i a_x = \eta_i$; let T' denote the expression for T when $\eta_1 \ldots \eta_n$ are introduced in place of $y_1 \ldots y_n$; then

$$-\Delta T' = \begin{vmatrix} a_{11}, & a_{12}, & \ldots a_{1n}, & \eta_1 \\ a_{21}, & a_{22}, & \ldots a_{2n}, & \eta_2 \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}, & a_{n2}, & \ldots a_{nn}, & \eta_n \\ \eta_1, & \eta_2, & \ldots \eta_n, & 0 \end{vmatrix} = D.$$

This is a well-known equation and need not be proved here. Now let δ denote a differential operator, acting only on the coefficients a_{ik}; let D_{ik}, Δ_{ik} denote the minors of D, Δ complementary to a_{ik}. Then we have $\delta D = \Sigma\Sigma D_{ik} \delta a_{ik}$.

Consider D_{11}; this is

$$\begin{vmatrix} a_{22}, & \ldots a_{2n}, & \eta_2 \\ \cdots & \cdots & \cdots \\ a_{n2}, & \ldots a_{nn}, & \eta_n \\ \eta_2, & \ldots \eta_n, & 0 \end{vmatrix};$$

reducing this by multiplying the columns by y_2 &c., subtracting from the last columns, and remembering the equations

$\eta_i = a_i a_y,\ T + \Sigma\eta_i y_i$, we get

$$D_{11} = \begin{vmatrix} a_{22}, & \dots a_{2n}, & a_{21} y_1 \\ \dots & \dots & \dots \\ a_{n2}, & \dots a_{nn}, & a_{n1} y_1 \\ \eta_2, & \dots \eta_n, & y_1 \eta_1 - T \end{vmatrix}$$

$$= -T\Delta_{11} + y_1 \begin{vmatrix} a_{21}, & \dots a_{2n}, & a_{21} \\ \dots & \dots & \dots \\ a_{n2}, & \dots a_{nn}, & a_{n1} \\ \eta_2, & \dots \eta_n, & \eta_1 \end{vmatrix}$$

$$= -T\Delta_{11} + y_1 \begin{vmatrix} a_{22}, & \dots a_{2n}, & a_{11} \\ \dots & \dots & \dots \\ a_{n2}, & \dots a_{nn}, & a_{n1} \\ a_{n1} y_1, & \dots a_{nn} y_1, & a_{11} y_1 \end{vmatrix}$$

$$= -T\Delta_{11} + y_1^2 \Delta.$$

In the same way we get generally

$$D_{ik} = -T\Delta_{ik} + y_i y_k \Delta;$$

and therefore

$$\delta D = -T\Sigma\Delta_{ik}\,\delta a_{ik} + \Delta\Sigma\delta a_{ik}\,y_i y_k = -T\delta\Delta + \Delta\delta T.$$

But $\qquad D = -T'\Delta,$

and therefore $\qquad \delta D = -\delta T'\Delta - T'\delta\Delta;$

but $\qquad T' = \ \ T;$

therefore $\qquad \delta T = -\delta T';$

and this gives at once the n equations

$$\frac{\partial T}{\partial q_i} = -\frac{\partial T'}{\partial q_i}.$$

We have also to show that

$$y_i = \tfrac{1}{2}\frac{\partial T}{\partial \eta_i};$$

we have
$$\tfrac{1}{2}\frac{\partial D}{\partial \eta_1} = \begin{vmatrix} a_{12}, & \ldots a_{1n}, & \eta_1 \\ a_{22}, & \ldots a_{2n}, & \eta_2 \\ \ldots & \ldots & \ldots \\ a_{n2}, & \ldots a_{nn}, & \eta_n \end{vmatrix}$$
$$= \begin{vmatrix} a_{12}, & \ldots a_{1n}, & ay \\ a_{21}, & \ldots a_{2n}, & a_{21}y_1 \\ \ldots & \ldots & \ldots \\ a_{n2}, & \ldots a_{nn}, & a_{n1}y_1 \end{vmatrix}$$
$$= -\Delta y_1;$$
therefore
$$\tfrac{1}{2}\frac{\partial T'}{\partial \eta_1} = y_1;$$

and in the same way we get the general equation
$$\frac{\partial T'}{\partial \eta_i} = 2y_i.$$

The above proof of the formulæ is not presented as being simpler than the investigations given in most books on dynamics, but it seemed worth while to give it in this form as showing how the greater part of theoretical dynamics depends on the fact that the kinetic energy is a quadric function of the velocities; besides, and this is of more importance, it seems advisable to keep the purely mathematical parts of a subject as free as possible from the taint of physical reasoning.

3. *An identity in the theory of matrices.*

Let A denote the matrix $\| a_{ik} \|$; then the identity in question is
$$\begin{vmatrix} a_{11}-A, & a_{12}, & \ldots & a_{1n} \\ a_{21}, & a_{22}-A, & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1}, & a_{n2}, & \ldots & a_{nn}-A \end{vmatrix} = 0.$$

The proof here given is merely an adaptation of the quaternion proof for the case of $n=3$, given in Prof. Tait's *Elementary Treatise on Quaternions* (p. 81).

Let A denote any matrix of order n; let A' be the reciprocal matrix, having as its elements the first minors of

det. A, so that $A' = \text{det}.A.A^{-1}$; let Δ denote det. A; let a be any scalar, and let $\text{det}.(A+a) = \Delta + c_1a + c_2a^2 + \ldots + a^n$. Then we have

$$\text{det}.(A+a).(A+a)^{-1} = (A+a)';$$

that is

$$(\Delta + c_1a + c_2a^2 + \ldots + a^n)(A+a)^{-1} = A' + \lambda_1 a + \lambda_2 a^2 + \ldots + a^{n-1},$$

where $\lambda_1, \lambda_2, \ldots$ are matrices to be eliminated in the sequel; now operate on both sides with $A + a$, remembering that $AA' = \Delta$, and equate the coefficients of the powers of a; we get

$$\begin{aligned} c_1 &= \lambda_1 A + A', \\ c_2 &= \lambda_2 A + \lambda_1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ c_{n-1} &= \quad A + \lambda_{n-2}, \end{aligned}$$

and therefore

$$c_1 A - c_2 A^2 + c_3 A_3 \ldots (-)^n c_{n-1} A^{n-1} = \Delta (-)^n A^n;$$

that is $\quad \Delta - c_1 A + c_2 A^2 \ldots (-)^n A^n = 0.$

This is Prof. Cayley's identity.

ON A DIFFERENTIAL EQUATION.

By *H. J. Sharpe, M.A.*

1. In pages 175 to 185 of vol. x of the *Messenger*, I discussed that solution of the equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + y(x + A) = 0 \ldots\ldots\ldots\ldots\ldots (1),$$

which ascends by powers of x and A, which does not involve $\log x$, and which is such that $y = 1$ when $x = 0$. I remarked that this equation arises in the investigation of the problem of the reflection of sound at the surface of a paraboloid, and that it is desirable to obtain methods for finding practically the values of y for any positive value of x, and for any value of y positive or negative. I have already partly supplied these methods, and it is the object of the present paper to supply cases not already discussed. Before doing so, however, I propose to make some observations on, and a necessary correction of, my former paper.

2. On page 180, I proved that

$$y = 1 + \frac{x^2}{(2\,!)^2}(A^2-1) + \frac{x^4}{(4\,!)^2}(A^4 - 14A^2 + 9) + \&c.$$

$$+ \frac{x^m}{m\,!}\{(m, m)A^m - (m, m-2)A^{m-2} + \&c. + (-1)^{\frac{1}{2}m} \times (m, 0)\} + \&c.$$

$$- xA - \frac{x^3}{(3\,!)^2}(A^3 - 5A) - \frac{x^5}{(5\,!)^2}(A^5 - 30A^3 + 89A) - \&c.$$

$$- \frac{x^m}{m\,!}\{(m, m)\,A^m - (m, m-2)\,A^{m-2} + \&c.$$

$$+ (-1)^{\frac{1}{2}(m-1)} \times (m, 1)\,A\} - \&c. \ldots\ldots\ldots (2),$$

m being supposed first even and then odd. The laws of formation of the quantities (m, m), $(m, m-2)$, &c. have been already given.

If we call the series (2) $(1 + a_1x + a_2x^2 + \&c.)$, we can readily shew that

$$(n+1)^2\,a_{n+1} + Aa_n + a_{n-1} = 0 \ldots\ldots\ldots\ldots (3).$$

I have also proved that

$$y = \frac{2}{\pi}\,\{\varepsilon^{\frac{1}{2}\pi A} + \varepsilon^{-\frac{1}{2}\pi A}\} \times \int_0^\infty \cos\left(x\frac{d}{s} + A\phi\right)\frac{d\phi}{s} \ldots\ldots (4),$$

where for brevity, s has been put for $\varepsilon^{\phi} + \varepsilon^{-\phi}$, and d for $\varepsilon^{\phi} - \varepsilon^{-\phi}$. I have already remarked, that if the integral in (4) be expanded in powers of x, we get a series which always converges, but if it be expanded in powers of A, we get a series which is convergent only when $A^2 < 1$. Perhaps a little more light can be thrown on this latter point. It may be proved that $\frac{1}{s}\cos\left(x\frac{d}{s}\right)$ and $\frac{1}{s}\sin\left(x\frac{d}{s}\right)$ can be expanded in converging series of odd powers of $\varepsilon^{-\phi}$. Imagine them so expanded. I shall not give the details, but we may write the integral

$$= \int_0^\infty [(x_1\varepsilon^{-\phi} + x_3\varepsilon^{-3\phi} + \&c.)\cos A\phi - (z_1\varepsilon^{-\phi} + z_3\varepsilon^{-3\phi} + \&c.)\sin A\phi]\,d\phi$$

$$= x_1\frac{1}{1+A^2} + x_3\frac{3}{3^2+A^2} + \&c. - z_1\frac{A}{1+A^2} - z_3\frac{A}{3^2+A^2} - \&c.$$

Now when we remember that

$$\varepsilon^{\frac{1}{2}\pi A} + \varepsilon^{-\frac{1}{2}\pi A} = 2\,(1+A^2)\left(1+\frac{A^2}{3^2}\right)\left(1+\frac{A^2}{5^2}\right)\ldots,$$

we see at once the part played by the factor in making (4 identical with (2). Of course x_1, x_2, &c. z_1, z_2, &c. ar functions of x.

3. On page 183, vol. x, I said that if we have the equatio

$$\mu y + \frac{dy}{dx} = 0 \text{ .... } (5),$$

y having the value (2) and μ being a constant, then whethe we consider (5) as an equation in A, x being given, or, as a equation in x, A being given, in either case the roots are a real. The first statement is undoubtedly true, and has bee proved. The second, which was given without proof, wa hastily arrived at, and is probably false. Oddly enougl however, a somewhat similar proposition, which it may b worth while to give is true. If we have the equation

$$\mu y + x\frac{dy}{dx} = 0 \text{ .... } (6),$$

then if A be small, the impossible roots of (6), if there a any, are small, and if A be large, the impossible roots of (6 if there are any, are large. The method of proof is simil to the well-known one of Fourier given in Art. 424 Todhunter's Laplace's Functions. Let $\alpha\rho$, $\beta\rho$ be two val of x satisfying (6).

In (1) let y become u when $\alpha\rho$ is put for x,

.... v .... $\beta\rho$ ....

Then from (1)

$$\rho\frac{d^2u}{d\rho^2} + \frac{du}{d\rho} + u(\alpha^2\rho + A\alpha) = 0,$$

$$\rho\frac{d^2v}{d\rho^2} + \frac{dv}{d\rho} + v(\beta^2\rho + A\beta) = 0,$$

therefore

$$\rho\left(v\frac{du}{d\rho} - u\frac{dv}{d\rho}\right) + (\alpha^2 - \beta^2)\int\rho uv\,d\rho + A(\alpha - \beta)\int uv\,d\rho = ($$

From (6), $\mu u + \rho\frac{du}{d\rho} = 0$, $\mu v + \rho\frac{dv}{d\rho} = 0$;

therefore $\quad v\frac{du}{d\rho} - u\frac{dv}{d\rho} = 0$ for all values of ρ.

therefore in (7), integrating from $\rho = 0$ up to any assigned value c, we have

$$(\alpha^2 - \beta^2) \int_0^c \rho uv \, d\rho + A (\alpha - \beta) \int_0^c uv \, d\rho = 0 \; \ldots\ldots \; (8).$$

Let, if possible, $\alpha\rho$, $\beta\rho$ be a pair of impossible roots of (6), say suppose α and $\beta = \cos\alpha \pm i \sin\alpha$. As it is in our power to suppose ρ positive, we see that in order for (8) to be satisfied, $\sin\alpha$ must have a contrary sign to A. And if A be small ρ must be small, and if A be large ρ must be large, for uv is of course necessarily positive, being the sum of two squares.

4. Of conrse (2) can be applied to the numerical calculation of y only when x and A are small or moderate. We proceed therefore to a transformation of y which is applicable, either to the case of x moderate and A large, or to the case of both large, but $\dfrac{x}{A}$ small. It is given on page 183, vol. X, thus

$$y = [1 - x^2 (\tfrac{1}{2} d^2 + \tfrac{1}{3} z d^3) + x^4 (\tfrac{3}{8} d^4 + \tfrac{11}{30} z d^5 + \tfrac{1}{18} z^2 d^6) - \&c.] \, X_1 \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; (9),$$

where $$z = Ax, \quad d = \frac{d}{dz},$$

and $$X_1 = 1 - z + \frac{z^2}{(2\,!)^2} - \frac{z^3}{(3\,!)^2} + \&c. = J_0 (2z^{\frac{1}{2}}) \; \ldots . \; (10).$$

(N.B. I have here corrected an obvious misprint. $\frac{1}{8}{}^1$ was put for $\frac{11}{30}$ on p. 183).

By differentiating (10) a few times, we readily get the following equations:

$$\left.\begin{array}{l} (1 + d + zd^2) \, X_1 = 0 \\ (d + 2d^2 + zd^3) \, X_1 = 0 \\ (d^2 + 3d^3 + zd^4) \, X_1 = 0 \end{array}\right\} \; \ldots\ldots\ldots\ldots\ldots \; (11).$$
$$\&c.$$

By use of these equations, it will be found that (9) becomes

$$[1 + x^2 (\tfrac{1}{3} d + \tfrac{1}{6} d^2) + x^4 (\tfrac{1}{18} d^2 + \tfrac{37}{90} d^3 + \tfrac{7}{360} d^4) + \&c.] \, X_1 \ldots (12).$$

5. It may be advisable to exhibit the law that connects the numerical coefficients $\frac{1}{3}$, $\frac{1}{6}$, &c. This cannot be got from my former paper. It may be found thus. Assume as a solution of (1) in accordance with what (12) suggests

$$(A+x) y = \frac{1}{x} (z+x^2) \, [Z_0 + x^2 Z_1 + x^4 Z_2 + \&c. + x^{2n} Z_n + \&c.] \; \ldots . \; (13),$$

where Z_0, Z_1, &c. are functions of z to be determined; therefore

$$\frac{dy}{dx}=\&c.+x^{2n-1}\left(Z_n'z+2nZ_n\right)+\&c. \quad\text{.......................... (14)},$$

$$x\frac{d^2y}{dx^2}=\&c.+x^{2n-1}\left[z\frac{d}{dz}\left(Z_n'z+2nZ_n\right)+(2n-1)\left(Z_n'z+2nZ_n\right)\right]+\&c. \quad\text{.................. (15)}.$$

Adding together (13), (14), (15), and equating to 0 the coefficient of x^{2n-1}, we get

$$z^2Z_n''(4n+1)zZ_n'+zZ_n+4n^2Z_n+Z_{n-1}=0 \text{.... (16)}.$$

Assume, in accordance with the suggestion of (12),

$$Z_n=\left[(n,n)\,d^n+(n,n+1)\,d^{n+1}+\&c.+(n,2n)\,d^{2n}\right]X_1,$$

where (n, n), &c. are functions of n to be determined. Z_{n-1} will be a similar function of $(n-1)$. Putting these in (16) and using equations (11) as before to get rid of the z and z^2, multiplying the powers of d, and finally equating to 0 the coefficients of d^{n-1}, $d^n \ldots d^{2n-1}$, we get

$$-3n(n,n)+(n-1,n-1)=0 \text{ .... (17)},$$

$$4n^2(n,n)-(4n+1)\{n(n,n)+(n,n+1)\}$$
$$+(n+1)n(n,n)+(n+2)(n,n+1)+(n-1,n)=0,$$

$$4n^2(n,n+1)-(4n+1)\{(n+1)(n,n+1)+(n,n+2)\}$$
$$+(n+2)(n+1)(n,n+1)+(n+3)(n,n+2)+(n-1,n+1)=0,$$

&c. &c.

$$4n^2(n,2n-1)-(4n+1)\{(2n-1)(n,2n-1)+(n,2n)\}$$
$$+2n(2n-1)(n,2n-1)+(2n+1)(n,2n)=0.$$

It will be found that the coefficients of d^{n-2} and d^{2n} vanish of themselves, which is a test of the truth of the method. By means of these equations, all the quantities (n, n), &c. can be successively determined.

6. It follows at once from (17), that

$$(n,n)=\frac{1}{3^n(n\,!)} \text{ .... (18)};$$

as z is large, we may take, as I took in my former paper

$$\pi X_1=\frac{\pi^{\frac{1}{2}}}{z^{\frac{1}{4}}}\cos\left(2z^{\frac{1}{2}}-\tfrac{1}{4}\pi\right) \text{ nearly .... (19)}.$$

If smaller terms are wanted, they can be got from Art. 408 of Todhunter's Laplace's Functions. If we differentiate (19) successively with regard to z, and consider only the most important term, we see that the differential coefficients of X_1 with regard to z descend by powers of $z^{\frac{1}{2}}$. Hence, looking at (12), we see that the series ascends on the whole by powers of $\dfrac{x^2}{(Ax)^{\frac{1}{2}}}$ or of $\left(\dfrac{x^3}{A}\right)^{\frac{1}{2}}$. If, therefore, x be moderate and A large, or if x and A both be large, but $\left(\dfrac{x^3}{A}\right)^{\frac{1}{2}}$ small, the series (12) converges rapidly. It is obvious that if x be large, A must be excessively larger for (12) to be useful. In this case writing down only the most important term, we have from (2) and (19)

$$y = \frac{\cos(2z^{\frac{1}{2}} - \frac{1}{4}\pi)}{z^{\frac{1}{4}}\pi^{\frac{1}{2}}} \quad\ldots\ldots\ldots\ldots\ldots\ldots (20),$$

where $z = Ax$.

7. We proceed now to another transformation of y, which is applicable either to the case of x large and A moderate, or to the case of both large, but $\dfrac{A}{x}$ small. In (1) put

$$y = \varepsilon^{ix}v + \varepsilon^{-ix}u \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21),$$

where $i = \sqrt{(-1)}$, and equate to 0 the coefficients of ε^{ix} and ε^{-ix} respectively. Then

$$x\frac{d^2v}{dx^2} + \frac{dv}{dx}(2ix + 1) + (A + i)\,v = 0 \quad\ldots\ldots (22),$$

u satisfies a similar equation with the sign of i changed.

Assume

$$(A + i)\,v = (A + i)\left[\alpha_m x^{-m} + \alpha_{m+1}x^{-(m+1)} + \&c.\right]\ldots(23);$$

therefore

$$(2ix + 1)\frac{dv}{dx} = (2ix + 1)\left[-m\alpha_m x^{-(m+1)} - (m+1)\,\alpha_{m+1}x^{-(m+2)} - \&c.\right],$$

$$x\frac{d^2v}{dx^2} = m\,(m + 1)\,\alpha_m x^{-(m+1)} + (m + 1)\,(m + 2)\,\alpha_{m-1}x^{-(m+2)} + \&c.$$

Whence

$$\left.\begin{array}{r}(A+i)\alpha_m - 2im\alpha_m = 0\\ (A+i)\alpha_{m+1} - 2i(m+1)\alpha_{m+1} - m\alpha_m + m\,(m+1)\alpha_m = 0\\ (A+i)\alpha_{m+2} - 2i(m+2)\alpha_{m+2} - (m+1)\alpha_{m+1} + (m+1)(m+2)\alpha_{m+1} = 0\\ \&c., \qquad\qquad \&c.\end{array}\right\}\ldots(24).$$

These equations are satisfied by putting

$$A+i-2im=0 \ldots\ldots\ldots\ldots\ldots\ldots(25),$$

and then α_{m+1}, &c. can be got as multiples of α_m.

From (25) m varies roughly as A, if A be large. Also

$$\frac{\alpha_{m+1}}{\alpha_m} \text{ varies roughly as } A^2,$$

$$\frac{\alpha_{m+2}}{\alpha_{m+1}} \ldots\ldots\ldots\ldots\ldots\ldots A^2, \text{ and so on.}$$

So that the series (23) for v virtually ascends by powers of $\frac{A^2}{x}$. If therefore x and A both be large, but $\frac{A^2}{x}$ small, (23) can be used for the numerical calculation of v. Similarly for u by changing the sign of i. In this case writing down only the most important term, we have

$$y=\alpha_m e^{ix}\times x^{-\frac{1}{2}+\frac{1}{2}iA}+\beta_m e^{-ix}\times x^{-\frac{1}{2}-\frac{1}{2}iA} \ldots\ldots\ldots(26).$$

This may be rationalized thus

$$y=\frac{B}{x^{\frac{1}{2}}}\cos\left(x+\tfrac{1}{2}A\log x\right)+\frac{C}{x^{\frac{1}{2}}}\sin\left(x+\tfrac{1}{2}A\log x\right)\ldots(27).$$

8. The constants B and C must be determined by comparing (27) with the definite integral solution (4). It will be advisable to change the form of the integral by putting $e^{\phi}=\cot\frac{1}{2}\theta$. It will then be found, putting for brevity y' for the integral in (4), that

$$y'=\tfrac{1}{2}\int_0^{\pi}\cos\left(x\cos\theta+A\log\cot\tfrac{1}{2}\theta\right)d\theta.$$

We will transform this integral by a method suggested by Art. 405 of Todhunter's *Laplace's Functions.*

Putting $\cos\theta=2\cos^2\frac{1}{2}\theta-1$, we have

$$y'=\tfrac{1}{2}\cos x\int_0^{\pi}\cos\left(2x\cos^2\tfrac{1}{2}\theta+A\log\cot\tfrac{1}{2}\theta\right)d\theta$$

$$+\tfrac{1}{2}\sin x\int_0^{\pi}\sin\left(2x\cos^2\tfrac{1}{2}\theta+A\log\cot\tfrac{1}{2}\theta\right)d\theta.$$

Considering for the present only the first of these integrals, and putting $2x\cos^2\frac{1}{2}\theta = t$, it becomes

$$=\tfrac{1}{4}\cos x\int_0^{2x}\cos\left(t+\tfrac{1}{2}A\log\frac{t}{2x-t}\right)\frac{dt}{t^{\frac{1}{2}}(2x-t)^{\frac{1}{2}}}.$$

Break up this integral into two parts, the first from 0 up to x, the second from x up to $2x$. In the second part put $2x - t = \tau$. Then the second part

$$=\tfrac{1}{4}\cos x\int_0^{x}\cos\left(2x-\tau-\tfrac{1}{2}A\log\frac{\tau}{2x-\tau}\right)\frac{d\tau}{\tau^{\frac{1}{2}}(2x-\tau)^{\frac{1}{2}}}$$

$$=\tfrac{1}{4}\cos x\cos 2x\int_0^{x}\cos\left(t+\tfrac{1}{2}A\log\frac{t}{2x-t}\right)\frac{dt}{t^{\frac{1}{2}}(2x-t)^{\frac{1}{2}}}$$

$$+\tfrac{1}{4}\cos x\sin 2x\int_0^{x}\sin\left(t+\tfrac{1}{2}A\log\frac{t}{2x-t}\right)\frac{dt}{t^{\frac{1}{2}}(2x-t)^{\frac{1}{2}}}.$$

Treating in a similar way the second integral in y', we shall finally get, collecting all the terms,

$$y'=\tfrac{1}{2}\cos x\int_0^{x}\cos\left(t+\tfrac{1}{2}A\log\frac{t}{2x-t}\right)\frac{dt}{t^{\frac{1}{2}}(2x-t)^{\frac{1}{2}}}$$

$$+\tfrac{1}{2}\sin x\int_0^{x}\sin\left(t+\tfrac{1}{2}A\log\frac{t}{2x-t}\right)\frac{dt}{t^{\frac{1}{2}}(2x-t)^{\frac{1}{2}}}\ldots(28).$$

This equation is exact and is true for all values of x and A. Suppose now either x large and A moderate, or suppose both large, but $\frac{A}{x}$ small. Then the multiplier

$$\frac{1}{(2x-t)^{\frac{1}{2}}}=\frac{1}{(2x)^{\frac{1}{2}}}\cdot\frac{1}{\left(1-\frac{t}{2x}\right)^{\frac{1}{2}}}.$$

In this expression "$\left(1-\frac{t}{2x}\right)^{\frac{1}{2}}$ may be taken as unity, so long as t is not large, and when t is large, the corresponding elements of the integrals are of no account, because then $\frac{1}{t^{\frac{1}{2}}}$ is small." Again,

$$\tfrac{1}{2}A\log\frac{t}{2x-t}=\tfrac{1}{2}A(\log t-\log 2x)+\tfrac{1}{2}A\left(\frac{t}{2x}+\&c.\right).$$

We can neglect the terms involving powers of t here. For where t is small, these terms are small, since $\frac{A}{x}$ is small, and when t is large the corresponding elements of the integral are small. Therefore in this case

$$y' = \tfrac{1}{2}\cos x \int_0^x \cos\{t + \tfrac{1}{2}A(\log t - \log 2x)\} \frac{dt}{t^{\frac{1}{2}}(2x)^{\frac{1}{2}}}$$
$$+ \tfrac{1}{2}\sin x \int_0^x \sin\{t + \tfrac{1}{2}A(\log t - \log 2x)\} \frac{dt}{t^{\frac{1}{2}}(2x)^{\frac{1}{2}}}.$$

As we are seeking the limit of y' when x becomes very large, we can put ∞ for x in the upper limit, and we have

$$y' = \frac{1}{2(2x)^{\frac{1}{2}}}\cos(x + \tfrac{1}{2}A\log x)\int_0^\infty \cos(t + \tfrac{1}{2}A\log\tfrac{1}{2}t)\frac{dt}{t^{\frac{1}{2}}}$$
$$+ \frac{1}{2(2x)^{\frac{1}{2}}}\sin(x + \tfrac{1}{2}A\log x)\int_0^\infty \sin(t + \tfrac{1}{2}A\log\tfrac{1}{2}t)\frac{dt}{t^{\frac{1}{2}}} \ldots (29).$$

Comparing this with (27) we see that B and C are determined. The definite integrals here are obviously convergent at the limit $t = \infty$, and by putting $\frac{1}{2}t = \psi^2$ a new variable, they can be readily shown to be convergent at the other limit. We thus get

$$y' = \frac{1}{x^{\frac{1}{2}}}\cos(x + \tfrac{1}{2}A\log x)\int_0^\infty \cos(2\psi^2 + A\log\psi)\,d\psi$$
$$+ \frac{1}{x^{\frac{1}{2}}}\sin(x + \tfrac{1}{2}A\log x)\int_0^\infty \sin(2\psi^2 + A\log\psi)\,d\psi \ldots (30).$$

And from (4) we must remember that y is equal to y' multiplied by

$$\frac{2}{\pi}(\varepsilon^{\frac{1}{4}\pi A} + \varepsilon^{-\frac{1}{4}\pi A}).$$

I am indebted to Professor Stokes for the remarkable theorem that if

$$\frac{K}{S} = \int_0^\infty \frac{\cos}{\sin}(2\psi^2 + A\log\psi)\,d\psi,$$

then $$K^2 + S^2 = \frac{\pi}{4(\varepsilon^{\pi A} + 1)}.$$

We see from (30) that this Theorem gives us a superior limit to the value of y'.

9. We have now considered two cases of x and A being large, the first where $\frac{x}{A}$ is small, the second where $\frac{x}{A}$ is large. There remains a third case, where $\frac{x}{A}$ is moderate, that is equal to a quantity μ, which is small compared with either x or A. To this case we now proceed. [But it must be carefully noted that it will turn out that the following investigation is applicable to two other cases also (1) when μ is large, and A moderate, (2) μ large and A large.]

Putting in (1) $x = A\mu$, and considering μ as a new variable in place of x, and putting, as we may, y' for y, (1) becomes

$$\mu\frac{d^2y'}{d\mu^2} + \frac{dy'}{d\mu} + A^2y'(\mu+1) = 0 \text{ ....(31).}$$

Also (4) becomes

$$y' = \int_0^\infty \cos\left(A\mu\frac{d}{s} + A\phi\right)\frac{d\phi}{s} \text{ ....(32).}$$

We want to find a solution of (31) for large values of A and moderate values of μ, such as to be equal to (32).

In (31) put $y' = \varepsilon^{iz}$ where z is a new function of μ. (This form is chosen only for analytical convenience; y' will ultimately be rationalized.)

(31) becomes

$$\mu\left\{i\frac{d^2z}{d\mu^2} - \left(\frac{dz}{d\mu}\right)^2\right\} + i\frac{dz}{d\mu} + A^2(\mu+1) = 0 \text{....(33).}$$

Assume, to satisfy this equation,

$$z = Az_{-1} + z_0 + \frac{z_1}{A} + \frac{z_2}{A^2} + \text{\&c.} \text{ ....(34),}$$

where z_{-1}, z_0, &c. are functions of μ to be determined. It will be found that (33) becomes

$$\mu i\left[Az''_{-1} + z''_0 + \frac{z''_1}{A} + \frac{z''_2}{A^2} + \text{\&c.}\right] + i\left[Az'_{-1} + z'_0 + \frac{z'_1}{A} + \frac{z'_2}{A^2} + \text{\&c.}\right]$$

$$-\mu\left[A^2(z'_{-1})^2 + (z'_0)^2 + \frac{(z'_1)^2}{A^2} + \text{\&c.} + 2Az'_{-1}z'_0 + 2z'_{-1}z'_0\right.$$

$$\left.+\frac{2}{A}(z'_{-1}z'_2 + z'_0z'_1) + \frac{2}{A^2}(z'_{-1}z'_3 + z'_0z'_2) + \text{\&c.}\right] + A^2(\mu+1) = 0,$$

where z'_{-1}, &c. means $\dfrac{dz_{-1}}{d\mu}$, &c. Equating to 0 the coefficients of A^2, A, &c., we get

$$\left.\begin{array}{r} -\mu(z'_{-1})^2+\mu+1=0 \\ \mu i z''_{-1}+iz'_{-1}-2\mu z'_{-1}z'_0=0 \\ \mu i z''_0+iz'_0-\mu(z'_0)^2-2\mu z'_{-1}z'_1=0 \\ \mu i z''_1+iz'_1-2\mu(z'_{-1}z'_2+z'_0z'_1)=0 \\ \mu i z''_2+iz'_2-\mu(z'_1)^2-2\mu(z'_{-1}z'_3+z'_0z'_2)=0 \\ \text{\&c.,} \qquad\qquad \text{\&c.} \end{array}\right\} \ldots\ldots(35).$$

10. These equations are completely integrable. Without attempting to exhibit the general form of the integration, which would be very complex, some important features of it may be exhibited. We can readily get from (35)

$$z'_{-1}=\frac{(\mu+1)^{\frac{1}{2}}}{\mu^{\frac{1}{2}}}, \qquad z'_0=\frac{i}{4}\,\frac{2\mu+1}{\mu(\mu+1)},$$

$$z'_1=\frac{1}{2^5}\,\frac{4\mu^2+1}{\mu^{\frac{3}{2}}(\mu+1)^{\frac{5}{2}}}, \qquad z'_2=\frac{i(-8\mu^3+4\mu^2-4\mu-1)}{2^6\mu^2(\mu+1)^4},$$

$$z'_3=\frac{-400\mu^4+448\mu^3-360\mu^2-128\mu-25}{2^{11}\mu^{\frac{5}{2}}(\mu+1)^{\frac{11}{2}}}, \quad \text{\&c.}$$

From these it is obvious to conjecture that beginning at z'_0, z'_n must take the form

$$z'_n=i^{\frac{1}{2}\{1+(-1)^n\}}\times\frac{M_{n+1}}{\mu^{\frac{1}{2}(n+2)}(\mu+1)^{\frac{1}{2}(3n+2)}}\ldots\ldots\ldots(36),$$

where M_{n+1} is of the form

$$(a_{n+1}\mu^{n+1}+a_n\mu^n+\text{\&c.}+a_0).$$

This conjecture may be verified by substituting in the most general of equations (35), and M_{n+1} can be determined in terms of M's with lower subscripts. The relation which determines M_{n+1} will be found to take three forms, (1) according as n is even, (2) odd and of the form $(4m+1)$, (3) odd and of the form $(4m+3)$. Now effecting the integrations we shall get

$$\left.\begin{array}{ll} z_{-1}=\mu^{\frac{1}{2}}(\mu+1)^{\frac{1}{2}}+\log\{\mu^{\frac{1}{2}}+(\mu+1)^{\frac{1}{2}}\}, & z_0=\dfrac{i}{4}\log\{\mu(\mu+1)\}, \\ z_1=\dfrac{-1}{48}\,\dfrac{4\mu^2+12\mu+3}{\mu^{\frac{1}{2}}(\mu+1)^{\frac{3}{2}}}, & z_2=\dfrac{i}{2^6}\,\dfrac{4\mu^2+1}{\mu(\mu+1)^3}, \\ \text{\&c.,} & \text{\&c.} \end{array}\right\}(37).$$

It may be deduced from equations (35) that of the quantities z_{-1}, z_0, &c. only the alternate ones, beginning from the first are real, and that only the alternate ones will involve the radicals $\mu^{\frac{1}{2}}$, $(\mu+1)^{\frac{1}{2}}$.

If we look at the series of quantities z_{-1}, z_1, z_2, &c. in (37), and for a moment suppose μ large, we shall see that

z_{-1} varies as μ nearly,
z_1 .... 1 ....,
z_2 .... μ^{-2}........, and so on,

so that the series of quantities appears to follow some such law as this. When μ is very large, z_1 vanishes compared with z_{-1}, z_3 compared with z_1, and so on. Similarly, we may conjecture that z_4 vanishes compared with z_2, z_6 with z_4, and so on. This conjecture may be verified. We can integrate (36) with regard to μ by putting $\mu = \tan^2\theta$ and integrating with regard to θ. We shall thus have to integrate a series of expressions like

$$\int s^{2m+1} \times c^{-2m-2n-\frac{1}{2}} \times d\theta \ldots\ldots\ldots\ldots\ldots\ldots (38),$$

where for shortness s has been put for $\sin\theta$ and c for $\cos\theta$, and where m has a series of values beginning at $-\frac{1}{2}(n+2)$ and increasing by unity up to $\frac{1}{2}n$. Thus the index of c is always negative, and the index of s varies from $-(n+1)$ to $(n+1)$. As long as m has such a value that the index of s in (38) is not -1, it can readily be shown by the use of the formula of reduction

$$\int s^p c^q d\theta = -\frac{s^{p+1}c^{q+1}}{q+1} + \frac{p+q+2}{q+1}\int s^p c^{q+2} d\theta,$$

that (38) will not involve a logarithm. It can involve one only when $2m+1=-1$, *i.e.* $m=-1$, and then n must be even. In this case (38) may involve

$$\int \frac{c\,d\theta}{s} \text{ or } \log\sin\theta \text{ or } \log\frac{\mu}{\mu+1}.$$

We thus see from (36) that even if z_n involves a logarithm it does so in the form $\log\frac{\mu}{\mu+1}$, which vanishes when μ becomes very large. In order then to find what z_n becomes when μ is very large, we can safely expand (36) in ascending power of $\frac{1}{\mu}$, choose out the chief term and integrate. We thus get z_n varies as $\frac{1}{\mu^n}$. We can thus safely assert that when μ is

large, z_n vanishes compared with z_{n-2} whether n be odd or even, and whatever value n have. We see thus that the series (34), though ultimately divergent, yet begins by converging rapidly, when either μ or A, or both of them are large. In (37) put $z_0 = iu_0$, $z_2 = iu_2$, &c. where u_0, u_2, &c. are found from (37), then from (34) we shall find that

$$y' = \varepsilon^{-\left\{u_0 + \frac{u_2}{A^2} + \frac{u_4}{A^4} + \&c.\right\}} \times \left[B \cos\left(Az_{-1} + \frac{z_1}{A} + \frac{z_3}{A^3} + \&c.\right)\right.$$

$$\left. + C \sin\left(Az_{-1} + \frac{z_1}{A} + \frac{z_3}{A^3} + \&c.\right)\right] \;\ldots\ldots\ldots\; (39),$$

where B and C are some functions of A that have to be determined.

11. This may be done thus. The investigations in the last article are applicable to the case of x large and A moderate, but this is one of the cases considered in Article 7. Accordingly in (39), suppose μ large and A moderate. Then writing down only the most important terms, we shall get from (37)

$$y' = \frac{1}{\mu^{\frac{1}{4}}(\mu+1)^{\frac{1}{4}}} \left[B \cos A \left\{\mu^{\frac{1}{2}}(\mu+1)^{\frac{1}{2}} + \log(\mu^{\frac{1}{2}} + \overline{\mu+1}^{\frac{1}{2}})\right\}\right.$$

$$\left. + C \sin A \left\{\mu^{\frac{1}{2}}(\mu+1)^{\frac{1}{2}} + \log(\mu^{\frac{1}{2}} + \overline{\mu+1}^{\frac{1}{2}})\right\}\right] \;\ldots\ldots\; (40).$$

For in (39), by the preceding article, z_1 vanishes compared with z_{-1}, z_3 compared with z_1, &c. Similarly u_4 compared witl u_2, &c. Supposing them in (40) μ large and A moderate, w get nearly

$$y' = \frac{1}{\mu^{\frac{1}{2}}}\left[B \cos A \left\{\mu + \tfrac{1}{2}\log\mu + \log 2\right\}\right.$$

$$\left. + C \sin A \left\{\mu + \tfrac{1}{2}\log\mu + \log 2\right\}\right] \;\ldots\ldots (41.)$$

Remembering that $\mu A = x$, the angle involved here is

$$= x + \frac{A}{2}\log x + A \log \frac{2}{A^{\frac{1}{2}}}.$$

Putting this in (41), we get

$$y' = \frac{A^{\frac{1}{2}}}{x^{\frac{1}{2}}}\left[\cos\left(x + \frac{A}{2}\log x\right)\left\{C \sin\left(A \log\frac{2}{A^{\frac{1}{2}}}\right) + B \cos\left(A \log\frac{2}{A^{\frac{1}{2}}}\right.\right.\right.$$

$$+ \sin\left(x + \frac{A}{2}\log x\right)\left\{C \cos\left(A \log\frac{2}{A^{\frac{1}{2}}}\right) - B \sin\left(A \log\frac{2}{A}\right.\right.$$

Comparing this with (30), we shall get

$$B=\frac{1}{A^{\frac{1}{2}}}\cos\left(A\log\frac{2}{A^{\frac{1}{2}}}\right)\int_0^\infty \cos(2\psi^2+A\log\psi)\,d\psi$$

$$-\frac{1}{A^{\frac{1}{2}}}\sin\left(A\log\frac{2}{A^{\frac{1}{2}}}\right)\int_0^\infty \sin(2\psi^2+A\log\psi)\,d\psi,$$

$$C=\frac{1}{A^{\frac{1}{2}}}\sin\left(A\log\frac{2}{A^{\frac{1}{2}}}\right)\int_0^\infty \cos(2\psi^2+A\log\psi)\,d\psi$$

$$+\frac{1}{A^{\frac{1}{2}}}\cos\left(A\log\frac{2}{A^{\frac{1}{2}}}\right)\int_0^\infty \sin(2\psi^2+A\log\psi)\,d\psi.$$

Thus B and C in (39) are determined.

NOTE ON TWISTS IN AN INFINITE ELASTIC SOLID.

By *Karl Pearson.*

1. The general equations for vibrations in an infinite elastic solid are with the usual notation

$$\left.\begin{aligned}(\lambda+\mu)\frac{d\theta}{dx}+\mu\nabla^2u&=\rho\frac{d^2u}{dt^2}\\(\lambda+\mu)\frac{d\theta}{dy}+\mu\nabla^2v&=\rho\frac{d^2v}{dt^2}\\(\lambda+\mu)\frac{d\theta}{dz}+\mu\nabla^2w&=\rho\frac{d^2w}{dt^2}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(\text{i}).$$

Let

$$\xi=\frac{dw}{dy}-\frac{dv}{dz},\quad \eta=\frac{du}{dz}-\frac{dw}{dx},\quad \zeta=\frac{dv}{dx}-\frac{du}{dy},$$

$$X=\frac{d\zeta}{dy}-\frac{d\eta}{dz},\quad Y=\frac{d\xi}{dz}-\frac{d\zeta}{dx},\quad Z=\frac{d\eta}{dx}-\frac{d\xi}{dy}.$$

Hence

$$\left.\begin{aligned}\mu\nabla^2\xi&=\rho\frac{d^2\xi}{dt^2}\\\mu\nabla^2\eta&=\rho\frac{d^2\eta}{dt^2}\\\mu\nabla^2\zeta&=\rho\frac{d^2\zeta}{dt^2}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(\text{ii}),$$

$$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = 0 \text{..................} \text{(iii)},$$

$$X = \frac{d\theta}{dx} - \nabla^2 u, \quad Y = \frac{d\theta}{dy} - \nabla^2 v, \quad Z = \frac{d\theta}{dz} - \nabla^2 w \text{....(iv)},$$

$$\left.\begin{aligned} \mu X &= (\lambda + 2\mu)\frac{d\theta}{dx} - \rho\frac{d^2 u}{dt^2} \\ \mu Y &= (\lambda + 2\mu)\frac{d\theta}{dy} - \rho\frac{d^2 v}{dt^2} \\ \mu Z &= (\lambda + 2\mu)\frac{d\theta}{dz} - \rho\frac{d^2 w}{dt^2} \end{aligned}\right\} \text{..............(v)}.$$

If θ or $\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0$, we have

$$\left.\begin{aligned} \mu . X &= -\rho\frac{d^2 u}{dt^2} \\ \mu . Y &= -\rho\frac{d^2 v}{dt^2} \\ \mu . Z &= -\rho\frac{d^2 w}{dt^2} \end{aligned}\right\} \text{..................(vi)}.$$

If u were vibrational and of the form $A \cos(nt + \alpha)$,

$$\mu X = n^2 \rho . u,$$

and the linear displacement u would be proportional to μX. If $n = \sqrt{\frac{\mu}{\rho}}$ we should have $X = u$.

Now θ represents the dilatation of any element of the elastic solid, and (ξ, η, ζ) represent the twist components of any element about the three axes of coordinates. Let (l, m, n) be the direction cosines of the axis of twist, and ω the resultant twist at any point of the elastic medium, then

$$\xi = l\omega, \quad \eta = m\omega, \quad \zeta = n\omega.$$

If we identify magnetic induction and twist, the electric current will be proportional to the quantity whose components are (X, Y, Z), and the above equations are identical with the general equations of the "Magnetic Field" (Clerk Maxwell, *Electricity and Magnetism*, § 616).

2. Multiply equation (iii) by dx, dy, dz, and integrate over any volume. Integrating by parts, we may write

$$\int (l'\xi + m'\eta + n'\zeta)\, dS = 0,$$

where S is the surface of the given volume, and (l', m', n') the direction cosines of the normal to dS.

Hence $$\int \omega \cos\theta dS = 0,$$

where $\theta =$ angle between axis of twist and the normal.

This equation expresses the fact that the total normal twist over any closed surface at any time is zero.

At every point of the elastic medium draw the lines given by the differential equations

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta} = \frac{ds}{\omega}.$$

We have then a system of curves which may be termed the *twist lines*. Round any elementary curve draw the twist lines, we thus form what may be termed a *twist filament*. Assume as the surface S the portion of a filament bounded by planes perpendicular to it at any two points. Then since there is no resolved normal twist at any point of the surface of a filament, and the area of the filament is supposed indefinitely small, we have

$$\alpha_1 \omega_1 = \alpha_2 \omega_2,$$

where α_1, α_2 are the normal sections of the filament, and ω_1, ω_2 the corresponding twists.

We find then:

(*a*) At every point of a twist filament the product of the twist and the section of the filament is a constant.

(*b*) No filament can end in the solid; it must either end at the boundary of the solid or else re-enter. Hence in the case of an infinite solid with no boundaries every twist filament re-enters, or we must have a series of twist *rings*; a twist filament which passes to infinity in two directions being looked upon merely as an infinitely large ring.

Since the quantities (X, Y, Z) satisfy the solenoidal condition, it is obvious that the above propositions hold likewise for them. (Every electric current re-enters and is proportional to the normal section of its 'filament.')

3. In problems on elastic solids we may be given one of two cases. The displacements are given, we have to determine the twist; or we may have the inverse case, the twist is given and the corresponding displacements deduced from it. In neither case is it necessary to consider the forces producing the state of twist or displacement.

The latter problem is analogous to that of vortex motion. Given the vortex-filaments to find the fluid velocity at every point.

From equations (iv) we obtain

$$u = \frac{1}{4\pi}\iiint \frac{X}{r}\, dx'\, dy'\, dz' - \frac{1}{4\pi}\iiint \frac{1}{r}\, \frac{d\theta}{dx'}\, dx'\, dy'\, dz'.$$

Now
$$X = \frac{d\zeta}{dy'} - \frac{d\eta}{dz'},$$

and if we integrate by parts, we find

$$u = \frac{1}{4\pi}\iint \frac{m\zeta - n\eta - l\theta}{r}\, dS + \frac{1}{4\pi}\iiint \theta\, \frac{d}{dx'}\left(\frac{1}{r}\right) dx'\, dy'\, dz'$$
$$- \frac{1}{4\pi}\iiint \left\{\zeta\, \frac{d}{dy'}\left(\frac{1}{r}\right) - \eta\, \frac{d}{dz'}\left(\frac{1}{r}\right)\right\} dx'\, dy'\, dz'.$$

Now the first surface integral vanishes if we take an infinite sphere and assume that the displacements at infinity are zero. Again

$$r^2 = (x - x')^2 + (y - y')^2 + (z - z')^2,$$

or
$$\frac{d}{dy'}\left(\frac{1}{r}\right) = -\frac{d}{dy}\left(\frac{1}{r}\right) \text{ \&c.}$$

Hence, putting $\chi = \frac{1}{4\pi}\iiint \frac{\theta}{r}\, dx'\, dy'\, dz'$,

and writing $dx'\, dy'\, dz' = dv$, we have for the value of w at (x, y, z)

$$u = \frac{1}{4\pi}\int\left\{\zeta\, \frac{d}{dy}\left(\frac{1}{r}\right) - \eta\, \frac{d}{dz}\left(\frac{1}{r}\right)\right\} dv + \frac{d\chi}{dx}.$$

Similarly

$$v = \frac{1}{4\pi}\int\left\{\xi\, \frac{d}{dz}\left(\frac{1}{r}\right) - \zeta\, \frac{d}{dx}\left(\frac{1}{r}\right)\right\} dv + \frac{d\chi}{dy}, \quad \ldots\text{(vii)}.$$

$$w = \frac{1}{4\pi}\int\left\{\eta\, \frac{d}{dx}\left(\frac{1}{r}\right) - \xi\, \frac{d}{dy}\left(\frac{1}{r}\right)\right\} dv + \frac{d\chi}{dz}$$

It would seem from equation (vii) that a given system of twist may be accompanied by any given system of dilation, as determined by χ. For our purposes we shall suppose that system chosen in which $\theta=0$. This supposition does not require that the elastic solid should be imcompressible. Any such system also connotes a displacement of determinable character at every point of the elastic medium. Again the *instantaneous* creation of a twist filament of finite magnitude is an impossibility, because every such filament pre-supposes a definite displacement at every point of the medium.

Equations (vii) may be written

$$\left.\begin{aligned} u &= \frac{dW}{dy} - \frac{dV}{dz} \\ v &= \frac{dU}{dz} - \frac{dW}{dx} \\ w &= \frac{dV}{dx} - \frac{dU}{dy} \end{aligned}\right\}, \text{ where } \left.\begin{aligned} U &= \frac{1}{4\pi}\int \frac{\xi}{r}\,dv \\ V &= \frac{1}{4\pi}\int \frac{\eta}{r}\,dv \\ W &= \frac{1}{4\pi}\int \frac{\zeta}{r}\,dv \end{aligned}\right\} \text{...(viii).}$$

These equations are precisely similar to those of vortex motion. We note at once that no twist filament produces displacement of a twist kind at any other point of the medium.

4. As a simple case showing the manner in which twist filaments produce displacement in an infinite elastic solid, we may treat the case of a single circular twist ring.

Take the plane of xy for that of the ring and let a normal to that plane at its centre be the axis of z. Then

$$\zeta=0, \quad \xi=\omega\sin\theta, \quad \eta=-\omega\cos\theta,$$

where θ is the angle between the radius of the rings to the element whose twist components are (ξ, η) and the axis of x. Any point on the top of the ring being thus displaced by the twist *towards* the axis z. Then

$$U=\frac{1}{4\pi}\int\frac{\omega\sin\theta}{r}\,dv, \quad V=-\frac{1}{4\pi}\int\frac{\omega\cos\theta}{r}\,dv.$$

Let a be the radius of the ring; therefore

$$dv = ad\theta\,\alpha.$$

while $\alpha\omega =$ a constant for the filament. Thus

$$U=\frac{\alpha\omega a}{4\pi}\int_0^{2\pi}\frac{\sin\theta}{r}\,d\theta, \quad V=-\frac{\alpha\omega a}{4\pi}\int_0^{2\pi}\frac{\cos\theta}{r}\,d\theta.$$

If $(\rho\cos\phi,\ \rho\sin\phi,\ z)$ be the coordinates of the point at which we wish to find the displacement, then

$$r^2 = z^2 + \rho^2 + a^2 - 2a\rho\cos(\phi-\theta),$$

and
$$U\sin\phi - V\cos\phi = \frac{\alpha\omega a}{4\pi}\int_0^{2\pi}\frac{\cos(\phi-\theta)}{r}\,d\theta$$
$$= \frac{\alpha\omega a}{4\pi}\int_0^{2\pi}\frac{\cos\theta}{r}\,d\theta,$$

where
$$r^2 = z^2 + \rho^2 + a^2 - 2a\rho\cos\theta,$$

which integral is independent of ϕ; hence we must have

$$U = P\sin\phi, \quad V = -P\cos\phi,$$

where
$$P = \frac{\alpha\omega a}{4\pi}\int_0^{2\pi}\frac{\cos\theta\,d\theta}{\sqrt{(z^2+a^2+\rho^2-2a\rho\cos\theta)}}.$$

We find at once

$$u = \cos\phi\frac{dP}{dz}, \quad v = \sin\phi\frac{dP}{dz},$$

$$w = \frac{dV}{dx} - \frac{dU}{dy} = -\left(\frac{P}{\rho} + \frac{dP}{d\rho}\right).$$

Hence we see the total displacement effect at any point due to the twist ring consists of (1) a radial displacement *towards* the axis of $z = -\frac{dP}{dz}$, and (2) a displacement perpendicular and *towards* the plane of the ring $= \frac{dP}{d\rho} + \frac{P}{\rho}$.

The expression for P in terms of the elliptic integrals F and E is

$$\frac{\alpha\omega}{4\pi}\sqrt{\frac{a}{\rho}}\left\{\left(k-\frac{2}{k}\right)F + \frac{2}{k}E\right\},$$

where
$$k = \text{modulus} = \frac{2\sqrt{(a\rho)}}{\sqrt{\{(a+\rho)^2+z^2\}}}.$$

5. The internal work done by a system of twists may be readily expressed.

With Lamé's notation the internal force on any element in the direction of the displacement on u

$$= -\left(\frac{dN_1}{dx} + \frac{dT_3}{dy} + \frac{dT_2}{dz}\right) = -(\lambda+\mu)\frac{d\theta}{dx} - \mu\nabla^2 u$$

$$= -(\lambda+2\mu)\frac{d\theta}{dx} + \mu X.$$

Hence the total internal work may be expressed by half the integral

$$-(\lambda+2\mu)\iiint\left(u\frac{d\theta}{dx}+v\frac{d\theta}{dy}+w\frac{d\theta}{dz}\right)dx\,dy\,dz$$
$$+\mu\iiint(Xu+Yv+Zw)\,dx\,dy\,dz.$$

Integrating by parts and supposing the solid to be in no state of strain at an infinite distance, we find

$$\text{work}=\tfrac{1}{2}(\lambda+2\mu)\iiint\theta^2\,dx\,dy\,dz$$
$$+\tfrac{1}{2}\mu\iiint\left\{\xi\left(\frac{dw}{dy}-\frac{dv}{dz}\right)+\eta\left(\frac{du}{dz}-\frac{dw}{dx}\right)+\zeta\left(\frac{dv}{dx}-\frac{du}{dy}\right)\right\}dx\,dy\,dz$$
$$=\tfrac{1}{2}(\lambda+2\mu)\int\theta^2 dv+\mu\iiint(\xi^2+\eta^2+\zeta^2)\,dx\,dy\,dz$$
$$=\tfrac{1}{2}\int\{(\lambda+2\mu)\,\theta^2+\mu\omega^2\}\,dv \qquad\text{(ix)},$$

a concise expression for the internal work of any infinite elastic solid. We note that the portion of this work due to twist

$$=\tfrac{1}{2}\mu\int\omega^2\,dv$$
$$=\tfrac{1}{2}\int(\mu\xi.\xi+\mu\eta.\eta+\mu\zeta.\zeta)\,dv$$
$$=\tfrac{1}{2}\int(\mu X.u+\mu Y.v+\mu Z.w)\,dv.$$

Hence we may say that the internal forces resisting twist are

$$\mu\xi,\quad \mu\eta,\quad \mu\zeta;$$

or, again, that $\quad \mu X,\quad \mu Y,\quad \mu Z$

are the displacement forces produced by twist. We have thus arrived at a physical meaning for the expressions X, Y, Z; they are proportional to the forces of displacement produced by a system of twist. If then a pure displacement wave enters a region of twist, the direction of displacement in the wave will be altered. For example, if u be the displacement supposed in the face of the wave and there be twist displacement forces μX, μY in the face of the wave parallel and perpendicular to the direction of u, then since

$$\left.\begin{aligned}\mu X&=-\rho\frac{d^2u}{dt^2}\\ \mu Y&=-\rho\frac{d^2v}{dt^2}\end{aligned}\right\},\qquad \left.\begin{aligned}u&=u_0-\frac{\mu}{\rho}\iint X\,(dt)^2\\ v&=\quad-\frac{\mu}{\rho}\iint Y\,(dt)^2\end{aligned}\right\};$$

or the direction of vibration is turned through an angle

$$=\frac{m\iint Y\,(dt)^2}{u_0.\rho-\mu\iint X\,(dt)^2}.$$

In the case of a twist filament the total internal work is given by

$$W = \mu \int \varpi \,.\, \omega a ds,$$

ds being an element of filament axis,

$$= \mu \omega a \int \omega ds,$$

$= \mu$. product of strength of filament $\times$ integral twist.

And generally internal work due to twists

$= \mu$. sum of product of strengths of twist filaments into their integral twists.

If the twist of the filament be uniform, the internal work

$$= \mu \omega^2 a \,.\, p,$$

$p =$ perimeter of axis of filament.

6. The following propositions may be noted:

(*a*) Consider any closed curve in the medium, then its total *tangential* displacement

$$= \int \left(u \frac{dx}{ds} + v \frac{dy}{ds} + w \frac{dz}{ds} \right) ds = \int (u\,dx + v\,dy + w\,dz)$$

$$= \int (l\xi + m\eta + n\zeta)\, dS,$$

where S is any surface bounded by the curve s, and (l, m, n) are the direction cosines of the normal to dS. It follows that the total tangential displacement of any closed curve drawn in a solid is equal to the total normal twist over any surface bounded by the curve.

(*b*) The total tangential twist round any curve in the solid

$$= \int \left(\xi \frac{dx}{ds} + \eta \frac{dy}{ds} + \zeta \frac{dz}{ds} \right) ds$$

$$= \int (\xi dx + \eta dy + \zeta dz)$$

$$= \int (lX + mY + nZ)\, dS;$$

or the total tangential twist round any closed curve is proportional to the total normal twist displacement force over any surface bounded by the curve.

Remembering equation (vi) and supposing our surface fixed in space, we may write

$$\int (lX + mY + nZ)\, dS = -\frac{\rho}{\mu} \frac{d^2}{dt^2} \int (lu + mv + nw)\, dS;$$

or the total tangential twist round any closed curve is proportional to the second differential coefficient with regard to the time of the total normal displacement of any surface bounded by the curve. In the simple case where u, v, w have a factor, only involving the time, of the form $\cos\left(\sqrt{\frac{\mu}{\rho}}\,t+\alpha\right)$, we may say that the total tangential twist is equal to the total normal displacement.

7. The question now arises, in any form of wave motion is it possible to propagate a system of twist filaments through an elastic medium in such a manner that the twist filaments while changing their position do not change their shape?

Some simple cases may be considered.

(*a*) Let the twist filaments be entirely in the face of a plane wave.

Take the axis of x as direction of the wave, then $\xi = 0$,

$$\frac{d\eta}{dy} + \frac{d\zeta}{dz} = 0,$$

or

$$\eta = \frac{d\phi}{dz} + F_1(z),$$

$$\zeta = -\frac{d\phi}{dy} + F_2(y),$$

where $F_1(z)$, $F_2(y)$ denote functions respectively of z and y only, and ϕ is a new arbitrary function. Again

$$\frac{d^2\eta}{dx^2} + \frac{d^2\eta}{dy^2} + \frac{d^2\eta}{dz^2} = \frac{\rho}{\mu}\,\frac{d^2\eta}{dt^2},$$

$$\frac{d^2\zeta}{dx^2} + \frac{d^2\zeta}{dy^2} + \frac{d^2\zeta}{dz^2} = \frac{\rho}{\mu}\,\frac{d^2\zeta}{dt^2},$$

Now we want the same twist filament state propogated in the direction of the axis of x. Hence we must take

$$\eta = \eta_0 . F\left(x - \sqrt{\frac{\mu}{\rho}}\,t\right),$$

$$\zeta = \zeta_0 . F\left(x - \sqrt{\frac{\mu}{\rho}}\,t\right),$$

where we consider the wave in the positive x direction only.

We then find

$$\frac{d^2\eta_0}{dy^2} - \frac{d^2\eta_0}{dz^2} = 0,$$

$$\frac{d^2\zeta_0}{dy^2} + \frac{d^2\zeta_0}{dz^2} = 0,$$

and η_0, ζ_0 of the form

$$\eta_0 = \frac{d\phi}{dz} + F_1(z),$$

$$\zeta_0 = -\frac{d\phi}{dy} + F_2(y).$$

Take $$\frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} + \frac{dF_1}{dz} - \frac{dF_2}{dy} = 0,$$

and the equations are satisfied.

Write for ϕ $$\int F_2\,dy - \int F_1\,dz + \phi,$$

and we find $$\eta_0 = \frac{d\phi}{dz}, \quad \zeta_0 = -\frac{d\phi}{dy},$$

where $$\frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0.$$

Form the curves given by $\phi = \text{constant}$. Then the twist at any point on one of these curves is given by

$$\eta_0 F = \phi_z F \quad \text{and} \quad \zeta_0 F = -\phi_y F,$$

and twist round the normal to $\phi = \text{constant}$

$$= \frac{\eta_0\phi_y + \zeta_0\phi_z}{\sqrt{(\phi_y^2 + \phi_z^2)}} F = 0,$$

twist round the tangent to $\phi = \text{constant}$

$$= \frac{\eta_0\phi_z - \zeta_0\phi_y}{\sqrt{(\phi_y^2 + \phi_z^2)}} F = \sqrt{(\phi_y^2 + \phi_z^2)}\, F,$$

or, the spaces marked off between the system of curves $\phi = \text{constant}$, are the twist filaments. We may then write

$$\omega = \sqrt{(\phi_y^2 + \phi_z^2)}\, F\left(x - \sqrt{\frac{\mu}{\rho}}\, t\right).$$

Now suppose $F(x)=0$ except when $x=a$ nearly, when it equals K. Then $\omega=0$ except when $x-\sqrt{\frac{\mu}{\rho}}\,t=a$ nearly, or $x=a+\sqrt{\frac{\mu}{\rho}}\,t$, and then it equals $\sqrt{(\phi_x^2+\phi_z^2)}\,K$.

We thus have a wave motion in which a plane of filaments of constant magnitude is propagated through an infinite elastic medium. The curves of twist at any point of the plane are determined by $\phi=0$, and the magnitude of the twist $=\sqrt{(\phi_y^2+\phi_z^2)}\,F$. The total (integral) twist of any twist curve $=F\int(\phi_y dz-\phi_z dy)=F\int d\psi=(\psi_1-\psi_0)\,F$, where ψ is the function conjugate to ϕ. If ψ is single valued there will be no integral twist. To calculate the effect of such a sheet of twist filaments in producing displacements at any point, and the corresponding twist displacement forces, we have to use the equations of articles (1) and (2). We have

$$X=0,$$

$$Y=\phi_y\cdot F'\left(x-\sqrt{\frac{\mu}{\rho}}\,t\right),$$

$$Z=\phi_z\cdot F'\left(x-\sqrt{\frac{\mu}{\rho}}\,t\right).$$

Hence $$\eta Y+\zeta Z=0,$$

or, *the twist force of displacement is perpendicular to the axis of twist.* (Cf. Clerk Maxwell's *Electricity and Magnetism*, Art. 791).

Again, if R be the resultant twist displacement force

$$R=\mu\frac{d\omega}{dx}=-\sqrt{(\mu\rho)}\,\frac{d\omega}{dt},$$

or, *the resultant twist force of displacement is proportional to the rate of change of twist.*

To find the displacements, we note that

$$\frac{dw}{dy}=\frac{dv}{dz},\quad w=\frac{d\chi}{dz},\quad v=\frac{d\chi}{dy},$$

$$\left.\begin{aligned}\eta&=\phi_z.F=\frac{du}{dz}-\frac{d^2\chi}{dx\,dz}\\ \zeta&=-\phi_y.F=\frac{d^2\chi}{dx\,dy}-\frac{du}{dy}\end{aligned}\right\}.$$

Hence $$u=\phi F+\frac{d\chi}{dx}.$$

Obviously, the really important part of these displacements is $u=\phi.F$, for by itself it is *sufficient* to produce the given

set of twists and twist displacement forces. It will be noted at once that this displacement is *normal* to the face of the wave, or perpendicular alike to the axis of twist and the twist displacement force. Hence, a plane wave of twist filaments may be generated by a set of displacements perpendicular to a plane (such, for example, as a set of molecules, *pulsating* or *oscillating* perpendicularly to that plane might produce; there is no occasion for a twist or torsion vibration in the bodies which set the wave going).

We might determine χ by the condition that

$$\theta = 0, \quad \text{or} \quad \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$$

We find $$\chi = \frac{1}{4\pi}\int \frac{\phi}{r}\frac{dF}{dx}\,dv = -\frac{1}{4\pi}\sqrt{\frac{\rho}{\mu}}\frac{d}{dt}\int\frac{F\phi}{r}\,dv.$$

Hitherto we have considered only the propagation of a plane sheet of twist filaments, but obviously by adding a number of such sheets together we can create a solitary wave of any kind or a succession of waves, the same sheet of filaments being repeated after an interval λ. We have only to take a series of functions $F_1(x)$, $F_2(x)$, $F_3(x)$, &c.; so that $F_1(x) = 0$, except when x lies between α and $\alpha + \delta\alpha$; $F_2(x) = 0$, except when x lies between $\alpha + \delta\alpha$ and $\alpha + 2\delta\alpha$, and so on. To each value of $F(x)$ we then give a peculiar value of ϕ. In practice, of course, each term should be expressed by an infinite series of sines or cosines of multiples of $\left(x - \sqrt{\frac{\mu}{\rho}}\,t\right)$.

It thus seems possible, that sets of plane twist rings may be propagated through an elastic medium without losing their individuality.

(*b*) Let the planes of the twist rings* be perpendicular to the plane front of the wave.

Take the axis of x as direction of the wave motion. We have $\eta = 0$, and we suppose ξ and ζ functions only of x and z. Hence

$$\frac{d\xi}{dx} + \frac{d\zeta}{dz} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (E),$$

$$\left.\begin{aligned} \frac{d^2\xi}{dx^2} + \frac{d^2\xi}{dz^2} &= \frac{\rho}{\mu}\frac{d^2\xi}{dt^2} \\ \frac{d^2\zeta}{dx^2} + \frac{d^2\zeta}{dz^2} &= \frac{\rho}{\mu}\frac{d^2\zeta}{dt^2} \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots (F).$$

* The twist rings in this article are really cylinders with generators parallel to the axis of y.

Now, if we assume $\xi = F\left(z, x - \sqrt{\frac{\mu}{\rho}}\, t\right)$ which corresponds to the movement of an *unchanged* ξ with velocity $\sqrt{\frac{\mu}{\rho}}$ in the direction of the axis of x, we find

$$\frac{d^2\xi}{dz^2} = 0,$$

similarly we must have $$\frac{d^2\zeta}{dz^2} = 0.$$

Hence from (E) $$\frac{d^2\xi}{dx\,dz} = 0,$$

or $$\xi = f_1(x) + Bz,$$

where B is an absolute constant. Therefore

$$\zeta = f_2'(x) - f_1'(x)\,.\,z,$$

substituting in the wave equations (F), we obtain the result

$$\xi = f_1\left(x - \sqrt{\frac{\mu}{\rho}}\, t\right) + Bz,$$

$$\zeta = f_2'\left(x - \sqrt{\frac{\mu}{\rho}}\, t\right) - f_1'\left(x - \sqrt{\frac{\mu}{\rho}}\, t\right) z.$$

Now these equations denote a twist becoming infinitely great with z; in order to avoid this inconsistency, we must take

$$B = 0, \quad f_1'\left(x - \sqrt{\frac{\mu}{\rho}}\, t\right) = 0.$$

Therefore $$\xi = C, \quad \zeta = f_2'\left(x - \sqrt{\frac{\mu}{\rho}}\, t\right).$$

The curves of twist are given by

$$\frac{dx}{\xi} = \frac{dz}{\zeta},$$

or $$Cz = f_2\left(x - \sqrt{\frac{\mu}{\rho}}\, t\right) + A.$$

We see then that the family of twist curves

$$Cz = f_2(x) + A$$

are propagated with a velocity $= \sqrt{\frac{\mu}{\rho}}$ parallel to the plane of xz.

For example, if ζ be represented by a single cosine term, the type of twist curve will be

$$z = \alpha + \beta \sin x,$$

and we must mass the whole plane of xz with curves of sines, and suppose the entire system to be propagated with velocity $\sqrt{\frac{\mu}{\rho}}$ from left to right (fig. 2) (which would represent the propagation of a series of twist filaments in the shape of the curve of sines whose planes are perpendicular to the face of the wave).

We find for the twist force of displacement

$$\mu X = 0, \quad \mu Y = -\mu f_2''\left(x - \sqrt{\frac{\mu}{\rho}}\, t\right), \quad \mu Z = 0,$$

in other words, the twist force of displacement is *perpendicular to the axis of twist*, and lies in the plane front of the wave.

To determine the displacement, we have

$$C = \frac{dw}{dy} - \frac{dv}{dz}, \quad 0 = \frac{du}{dz} - \frac{dw}{dx},$$

$$\zeta = \frac{df_2}{dx} = \frac{dv}{dx} - \frac{du}{dy}.$$

Now u and w are not functions of y. Hence, we find

$$u = \frac{d\chi}{dx}, \quad w = \frac{d\chi}{dz},$$

$$v = f_2\left(x - \sqrt{\frac{\mu}{\rho}}\, t\right) - Cz.$$

The characteristic portion of this displacement is v. We find

$$Y = -\frac{\rho}{\mu} \cdot \frac{d^2 v}{dt^2},$$

as equation (vi) would have led us expect. The displacement is parallel to the front of the wave, and perpendicular to the axis of twist.

Other simple forms of the twist curves in this particular case of wave motion, are straight lines inclined at various angles to the wave face.

(c). If we consider the most general case in which the

twist filaments lie in planes perpendicular to the face of the wave, we have from (E)

$$\xi = -\phi_z, \quad \zeta = \phi_x,$$

and ξ and ζ not functions of y, while $\eta = 0$.

Hence $$X = Z = 0,$$

and $$\mu Y = -\mu(\phi_{zz} + \phi_{xx}),$$

or, the twist displacement force is perpendicular to the axis of twist and parallel to the plane face of the wave.

The equations of wave motion (F) will be satisfied if

$$\phi_{xx} + \phi_{zz} = \frac{\rho}{\mu}\phi_{tt}.$$

Now assume ϕ to be a function of $x - n\sqrt{\frac{\mu}{\rho}}\,t$, then

$$\phi_{xx}(1 - n^2) + \phi_{zz} = 0.$$

Hence, if $\phi(z', x')$ be a solution of the equation

$$\frac{d^2\phi}{dx'^2} + \frac{d^2\phi}{dz'^2} = 0,$$

$$\phi\left\{z\sqrt{(1-n^2)}, \quad x - n\sqrt{\frac{\mu}{\rho}}\,t\right\},$$

is a function satisfying our equations.

The twist displacement force then becomes

$$\mu Y = -\mu n^2 \phi_{xx},$$

while the displacement itself is given by

$$u = \frac{d\chi}{dx}, \quad w = \frac{d\chi}{dz}, \quad v = \phi,$$

the last component being the characteristic portion.

Now in order to obtain any chosen set of twist curves, we should in general have to give a number of different values to n; hence we note, that various elements of twist would be propagated in the direction of the axis of x with different velocities.

For example, supposing the twist to vanish at infinity, we may take

$$\phi = \Sigma e^{-pz\sqrt{(1-n^2)}}\left\{\alpha_p \cos p\left(x - n\sqrt{\frac{\mu}{\rho}}\,t\right) + \beta_p \sin p\left(x - n\sqrt{\frac{\mu}{\rho}}\,t\right)\right\}.$$

Hence, when $t=0$,

$$\phi = \Sigma e^{-pz\sqrt{(1-n^2)}}(\alpha_p \cos px + \beta_p \sin px).$$

We may then determine α, β, p, n to suit some given state of twist filaments. In general, after time t we shall *not* have the same state merely displaced.

Thus, let the state be given by two terms

$$\alpha e^{-qz} \cos mx + \alpha' e^{-q'z} \cos m'x.$$

Then after time t the twist curves will be given by

$$\alpha e^{-qz} \cos m \left[x - \sqrt{\left\{\left(1-\frac{q^2}{m^2}\right)\frac{\mu}{\rho}\right\}} t\right] + \alpha' e^{-q'z} \cos m' \left[x - \sqrt{\left\{\left(1-\frac{q'^2}{m'^2}\right)\frac{\mu}{\rho}\right\}} t\right].$$

We see that the two elements have been propagated at different rates, namely

$$\sqrt{\left\{\left(1-\frac{q^2}{m^2}\right)\frac{\mu}{\rho}\right\}} \quad \text{and} \quad \sqrt{\left\{\left(1-\frac{q'^2}{m'^2}\right)\frac{\mu}{\rho}\right\}},$$

only when $\frac{q}{m} = \frac{q'}{m'}$ will these be the same.

In the case of a single term

$$\alpha e^{-qz} \cos m \left\{x - \sqrt{\left(1-\frac{q^2}{m^2}\right)\frac{\mu}{\rho}}\, t\right\},$$

we note that the system does not lose its individuality, but its rate of advance through the medium $= \sqrt{\left\{\left(1-\frac{q^2}{m^2}\right)\frac{\mu}{\rho}\right\}}$ is different from the usual velocity of twist waves $\sqrt{\frac{\mu}{\rho}}$ in the elastic medium. Obviously we shall not interfere with our solution if we add to ϕ a term of the form $-Cz$, where C is an absolute constant, we then have

$$\phi = f\left\{z\sqrt{(1-n^2)},\quad x - n\sqrt{\frac{\mu}{\rho}}\, t\right\} - Cz.$$

Comparing this with the result of (b) where

$$\phi = f\left(x - \sqrt{\frac{\mu}{\rho}}\, t\right) - Cz.$$

We note:

(i) That if a system of twist curves in planes perpendicular to the face of the wave by entering a different medium, or in any other manner has its twist components made functions of their situation in the face of the wave (functions of z, and not only of x), then elements of twist will be propagated with different velocities, and the stable form of the twist curves will be destroyed. (We have something which might be termed a "scattering of twist").

(ii) In the simple case where ϕ is of the form

$$\phi = A \sin m\left(x - \sqrt{\frac{\mu}{\rho}}\, t\right) - Cz$$

in the ordinary medium, and upon the twist curves entering a different medium, or in some other manner becomes

$$\phi = Ae^{-qz} \cos m\left[x - \sqrt{\left\{\left(1 - \frac{q^2}{m^2}\right)\frac{\mu}{\rho}\right\}}\, t\right] - Cz,$$

there is still a stability in the twist curves, their shape it is true has been changed, but they are propagated with the velocity $\sqrt{\left\{\left(1 - \frac{q^2}{m^2}\right)\frac{\mu}{\rho}\right\}}$ common to them all, *i.e.* there is a *retardation* in the rate of wave motion.

March 3, 1883.

ON A PECULIAR DEVELOPMENT OF A SPECIAL DETERMINANT OF THE SIXTH ORDER.

By *Thomas Muir, M.A., F.R.S.E.*

§1. The determinant in question is

$$\begin{vmatrix} a_1 & a_2 & a_3 & Aa_1 & Aa_2 & Aa_3 \\ b_1 & b_2 & b_3 & Bb_1 & Bb_2 & Bb_3 \\ c_1 & c_2 & c_3 & Cc_1 & Cc_2 & Cc_3 \\ d_1 & d_2 & d_3 & Dd_1 & Dd_2 & Dd_3 \\ e_1 & e_2 & e_3 & Ee_1 & Ee_2 & Ee_3 \\ f_1 & f_2 & f_3 & Ff_1 & Ff_2 & Ff_3 \end{vmatrix}, \text{ or } \Delta \text{ say;}$$

the capital letters and the corresponding small letters having no relation to one another.

§ 2. Expressing it as an aggregate of products of complementary minors formed from the first and last three columns, and combining the products in pairs, we have

$$\begin{aligned}
\Delta = \;& |a_1b_2c_3|\,|d_1e_2f_3|\,(DEF-ABC)\\
-\;& |a_1b_2d_3|\,|c_1e_2f_3|\,(CEF-ABD)\\
+\;& |a_1b_2e_3|\,|c_1d_2f_3|\,(CDF-ABE)\\
-\;& |a_1b_2f_3|\,|c_1d_2e_3|\,(CDE-ABF)\\
+\;& |a_1c_2d_3|\,|b_1e_2f_3|\,(BEF-ACD)\\
-\;& |a_1c_2e_3|\,|b_1d_2f_3|\,(BDF-ACE)\\
+\;& |a_1c_2f_3|\,|b_1d_2e_3|\,(BDE-ACF)\\
+\;& |a_1d_2e_3|\,|b_1c_2f_3|\,(BCF-ADE)\\
-\;& |a_1d_2f_3|\,|b_1c_2e_3|\,(BCE-ADF)\\
+\;& |a_1e_2f_3|\,|b_1c_2d_3|\,(BCD-AEF),
\end{aligned}$$

or, say

$$\Delta=\pi_1\,(DEF-ABC)-\pi_2(CEF-ABD)+\ldots+\pi_{10}(BCD-AEF),$$

the ten products of pairs of determinants being denoted by $\pi_1, \pi_2, \ldots, \pi_{10}$ in order.

Now by a theorem of Sylvester's (*Phil. Mag.*, 4th ser. II. pp. 142–145),

$$|a_1b_2c_3|\,|d_1e_2f_3| = |a_1d_2c_3|\,|b_1e_2f_3| + |a_1e_2c_3|\,|d_1b_2f_3| + |a_1f_2c_3|\,|d_1e_2b_3|,$$

i. e. $\qquad \pi_1 = -\pi_5 + \pi_6 - \pi_7.$

Similarly

$$|a_1b_2d_3|\,|c_1e_2f_3| = |a_1c_2d_3|\,|b_1e_2f_3| + |a_1e_2d_3|\,|c_1b_2f_3| + |a_1f_2d_3|\,|c_1e_2b_3|,$$

i. e. $\qquad \pi_2 = \pi_5 + \pi_8 - \pi_9.$

Also

$$|a_1b_2e_3|\,|c_1d_2f_3| = |a_1c_2e_3|\,|b_1d_2f_3| + |a_1d_2e_3|\,|c_1b_2f_3| + |a_1f_2e_3|\,|c_1d_2b_3|,$$

i. e. $\qquad \pi_3 = \pi_6 - \pi_8 - \pi_{10},$

and

$$|a_1b_2f_3|\,|c_1d_2e_3| = |a_1c_2f_3|\,|b_1d_2e_3| + |a_1d_2f_3|\,|c_1b_2e_3| + |a_1e_2f_3|\,|c_1d_2b_3|,$$

i. e. $\qquad \pi_4 = \pi_7 - \pi_9 + \pi_{10}.$

Using these identities to eliminate $\pi_1, \pi_2, \pi_3, \pi_4$ from the expansions obtained for Δ, we have

$$\begin{aligned}\Delta = &\ (-\pi_5+\pi_6-\pi_7)(DEF-ABC)\\ &+(-\pi_5-\pi_8+\pi_9)(CEF-ABD)\\ &+\ (\pi_6-\pi_8-\pi_{10})(CDF-ABE)\\ &+(-\pi_7+\pi_9-\pi_{10})(CDE-ABF),\\ &+\pi_5\ (BEF-ACD)-\pi_6(BDF-ACE)\\ &+\pi_7\ (BDE-ACF)+\pi_8(BCF-ADE)\\ &-\pi_9\ (BCE-ADF)+\pi_{10}(BCD-AEF),\\ =&-\pi_5\ (-ABC+DEF-ABD+CEF+ACD-BEF)\\ &+\pi_6\ (-ABC+DEF-ABE+CDF+ACE-BDF)\\ &-\pi_7\ (-ABC+DEF-ABF+CDE+ACF-BDE)\\ &-\pi_8\ (-ABD+CEF-ABE+CDF+ADE-BCF)\\ &+\pi_9\ (-ABD+CEF-ABF+CDE+ADF-BCE)\\ &-\pi_{10}(-ABE+CDF-ABF+CDE+AEF-BCD).\end{aligned}$$

But, again by Sylvester's theorem

$$|a_1b_2c_3|\,|d_1e_2f_3| = |b_1c_2d_3|\,|a_1e_2f_3| + |e_1b_2c_3|\,|d_1a_2f_3| + |f_1b_2c_3|\,|d_1e_2a_3|$$

i.e. $\qquad \pi_1 = \quad \pi_{10} \quad - \quad \pi_9 \quad + \quad \pi_8,$

and we have already had

$$\pi_1 = \quad -\pi_5 \quad + \quad \pi_6 \quad - \quad \pi_7.$$

Consequently, any one of $\pi_5, \pi_6, \pi_7, \pi_8, \pi_9, \pi_{10}$ can be expressed in terms of the remaining five, and so be eliminated from the above expression for Δ. Selecting π_5, we insert instead of it $\pi_6-\pi_7-\pi_8+\pi_9-\pi_{10}$, and thus obtain

$$\begin{aligned}\Delta = &\ \pi_6\ (-ABE+CDF+ACE-BDF+ABD-CEF-ACD+BEF)\\ &-\pi_7\ (-ABF+CDE+ACF-BDE+ABD-CEF-ACD+BEF)\\ &-\pi_8\ (-ABE+CDF+ADE-BCF+ABC-DEF-ACD+BEF)\\ &+\pi_9\ (-ABF+CDE+ADF-BCE+ABC-DEF-ACD+BEF)\\ &-\pi_{10}(-ABE+CDF-ABF+CDE+AEF-BCD\\ &\qquad\qquad +ABC-DEF+ABD-CEF-ACD+BEF).\end{aligned}$$

But the coefficients here of $\pi_6, \pi_7, \pi_8, \pi_9$ can be put in the forms

$$\begin{aligned}&-(F-A)(E-D)(C-B),\\ &-(F-D)(E-A)(C-B),\\ &-(F-A)(E-C)(D-B),\\ &-(F-C)(E-A)(D-B);\end{aligned}$$

and the coefficient of π_{10} equals

$$-(F-A)(E-D)(C-B)-(F-C)(E-B)(D-A),$$

or $$-(F-D)(E-A)(C-B)-(F-B)(E-C)(D-A),$$

or $$-(F-C)(E-A)(D-B)-(F-B)(E-D)(C-A),$$

or $$\begin{vmatrix} 1 & A+B & AB \\ 1 & C+D & CD \\ 1 & E+F & EF \end{vmatrix}.$$

Hence, we have the result

$$\begin{aligned}\Delta = -\pi_6\ &(F-A)(E-D)(C-B)+\pi_7(F-D)(E-A)(C-B)\\ +\pi_8\ &(F-A)(E-C)(D-B)-\pi_9(F-C)(E-A)(D-B)\\ +\pi_{10}\ &\{(F-A)(E-D)(C-B)+\ (F-C)(E-B)(D-A)\}\end{aligned}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(I)}.$$

§ 3. It is impossible to have fewer than *five* of the π-products appearing in the result; but as we have the selection of the five, there would appear to be a possibility of so choosing them, that the coefficients of *all* the five might be alike in form.

Now any one of the π-products can be expressed in six ways as an aggregate of three of the others. For example, we have

$$\begin{aligned}\pi_6 = \pi_1+\pi_5+\pi_7, \qquad &= \pi_1+\pi_3+\pi_9,\\ \pi_3+\pi_8+\pi_{10}, \qquad &\quad \pi_5+\pi_8+\pi_4,\\ \pi_9+\pi_4+\pi_2, \qquad &\quad \pi_7+\pi_{10}+\pi_2,\end{aligned}$$

where it has to be noted, that as the sum of the first terms of the first three expressions gives the first of the last three expressions, the sum of the second terms the second of the last three expressions, and the sum of the third terms the third of the last three expressions, it follows that any one of the six expressions is deducible from the remaining five. Only two of the six are applicable for our present purpose, since only two of them involve π_{10}, and so offer the possibility of doing away with the irregularity in the coefficient of that product. Taking the first of the two, viz. $\pi_3+\pi_8+\pi_{10}$, and substituting

$$(F-A)(E-B)(D-C)$$

for $(F-A)(E-C)(D-B)-(F-A)(E-D)(C-B)$,

we find as desired

$$\begin{aligned}\Delta = -\pi_2\,(F-A)(E-D)(C-B) &+ \pi_7(F-D)(E-A)(C-B)\\ +\pi_3\,(F-A)(E-B)(D-C) &- \pi_9(F-C)(E-A)(D-B)\\ +\pi_{10}(F-C)(E-B)(D-A) &\ldots\ldots\ldots\ldots\ldots\ldots \text{(II)}.\end{aligned}$$

Taking the other equivalent of π_6, viz. $\pi_7 + \pi_{10} + \pi_2$, we have

$$\begin{aligned}\Delta = -\pi_2\,(F-A)(E-D)(C-B) &+ \pi_7(F-E)(D-A)(C-B)\\ +\pi_3\,(F-A)(E-C)(D-B) &- \pi_9(F-C)(E-A)(D-B)\\ +\pi_{10}(F-C)(E-B)(D-A) &\ldots\ldots\ldots\ldots\ldots\ldots \text{(II}'\text{)}.\end{aligned}$$

§4. Of course by transposing rows of Δ and then applying (II) we might derive all the other like developments. But they can also be obtained from (II) as (II) was obtained from (I). Thus, supposing we wish a development similar to that in (II) but involving some other product than π_{10}, we first deduce from the above six equivalents of π_6 the six equivalents of π_{10}. These are

$$\begin{array}{ll} \pi_5 - \pi_2 + \pi_1, & \pi_5 - \pi \ + \pi_4, \\ -\pi_3 + \pi \ - \pi_8, & -\pi_2 + \pi_6 - \pi_7, \\ \pi_4 - \pi_7 + \pi_9, & \pi_1 - \pi_3 + \pi_9. \end{array}$$

Four of them would evidently introduce *two* new products instead of one, and are thus ineligible. Of the remaining two, $\pi_4 - \pi_7 + \pi_9$ and $\pi_1 - \pi_3 + \pi_9$, the one involving π_7 is the less likely, as the coefficients of π_7 and π_{10} in (II) have no common factor. Taking therefore the other, and writing

$$-(F-D)(E-B)(C-A)$$

for $(F-A)(E-B)(D-C)-(F-C)(E-B)(D-A)$,

and $(F-C)(E-D)(B-A)$,

for $(F-C)(E-B)(D-A)-(F-C)(E-A)(D-B)$,

we have

$$\begin{aligned}\Delta = \ \ \pi_1(F-C)(E-B)(D-A) &- \pi_2(F-A)(E-D)(C-B)\\ +\pi_7(F-D)(E-A)(C-B) &- \pi_3(F-D)(E-B)(C-A)\\ -\pi_9(F-C)(E-D)(B-A) &\ldots\ldots\ldots\ldots\ldots\ldots \text{(II}''\text{)}.\end{aligned}$$

§5. Of determinants of this kind, I find on examination that only one has been considered hitherto, viz.

$$\begin{vmatrix} 1 & a & A & aA \\ 1 & b & B & bB \\ 1 & c & C & cC \\ 1 & d & D & dD \end{vmatrix}, \text{ or } \Delta_4 \text{ say,}$$

which is shown (*Giornale di Matematica*, II, pp. 315, 316) to be equal to

$$(b-a)(d-c)(C-B)(D-A)-(d-a)(c-b)(B-A)(D-C).$$

The method followed above gives this result with much simplicity. We have first

$$\Delta_4 = (b-a)(d-c)\,CD-(c-a)(d-b)\,BD+(d-a)(c-b)\,BC$$
$$+(b-a)(d-c)\,AB-(c-a)(d-b)\,AC+(d-a)(c-b)\,AD.$$

Then putting in this $A=B=C=D=1$, we have the relation corresponding to the relations between the π-products in the preceding, viz.:

$$(c-a)(d-b)=(b-a)(d-c)+(d-a)(c-b).$$

Hence, eliminating $(c-a)(d-b)$, we have

$$\Delta_4 = (b-a)(d-c)(CD+AB-BD-AC)$$
$$+(d-a)(c-b)(BC+AD-BD-AC),$$

which is virtually the desired result.

§6. Similarly

$$\begin{vmatrix} 1 & a & aA \\ 1 & b & bB \\ 1 & c & cC \end{vmatrix} = Aa(c-b)-Bb(c-a)+Cc(b-a),$$

and

$$0 = a(c-b)-b(c-a)+c(b-a);$$

therefore

$$\begin{vmatrix} 1 & a & aA \\ 1 & b & bB \\ 1 & c & cC \end{vmatrix} = a(c-b)(A-B)+c(b-a)(C-B):$$

and from this we readily have another result given in the 'Giornale,' viz.:

$$\begin{vmatrix} 1 & a+A & aA \\ 1 & b+B & bB \\ 1 & c+C & cC \end{vmatrix} = (c-b)(B-A)(C-a)+(C-B)(b-a)(c-A).$$

§ 7. These latter identities point to the existence, in the case of certain determinants of *odd* order, of a development analogous to that found above for our special determinant of the sixth order.

Taking the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 & Aa_1 & Aa_2 \\ b_1 & b_2 & b_3 & Bb_1 & Bb_2 \\ c_1 & c_2 & c_3 & Cc_1 & Cc_2 \\ d_1 & d_2 & d_3 & Dd_1 & Dd_2 \\ e_1 & e_2 & e_3 & Ee_1 & Ee_2 \end{vmatrix}, \text{ or } \Delta_5 \text{ say};$$

and expanding as before, we have

$$\begin{aligned} \Delta_5 = & \ |a_1b_2c_3|\,|d_1e_2|\,DE - |a_1b_2d_3|\,|c_1e_2|\,CE \\ & + |a_1b_2e_3|\,|c_1d_2|\,CD + |a_1c_2d_3|\,|b_1e_2|\,BE \\ & - |a_1c_2e_3|\,|b_1d_2|\,BD + |a_1d_2e_3|\,|b_1c_2|\,BC \\ & - |b_1c_2d_3|\,|a_1e_2|\,AE + |b_1c_2e_3|\,|a_1d_2|\,AD \\ & - |b_1d_2e_3|\,|a_1c_2|\,AC + |c_1d_2e_3|\,|a_1b_2|\,AB, \\ = & \ \omega_1 DE - \omega_2 CE + \ldots + \omega_{10} AB; \end{aligned}$$

$\omega_1, \omega_2, \ldots, \omega_{10}$ being written for the ten products of complementary minors. These products are connected by means of a theorem analogous to Sylvester's theorem above quoted. For the particular case at present needed the new theorem is readily established in the manner Professor Cayley has established the other.* It is evident that

$$\begin{vmatrix} a_1 & a_2 & a_3 & . & . \\ b_1 & b_2 & b_3 & . & . \\ c_1 & c_2 & c_3 & . & . \\ d_1 & d_2 & d_3 & d_1 & d_2 \\ e_1 & e_2 & e_3 & e_1 & e_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & . & . \\ b_1 & b_2 & b_3 & . & . \\ c_1 & c_2 & c_3 & . & . \\ . & . & d_3 & d_1 & d_2 \\ . & . & e_3 & e_1 & e_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \\ c_1 & c_2 & c_3 & c_1 & c_2 \\ . & . & d_3 & d_1 & d_2 \\ . & . & e_3 & e_1 & e_2 \end{vmatrix};$$

* *Quarterly Journal of Mathematics* xv. pp. 55–57.

and therefore, from expressing the first and third of these determinants in terms of products of complementary minors, we have

$$| a_1b_2c_3 \,||\, d_1e_2 |$$

$$= | a_1b_2 \,||\, c_1d_2e_3 | - | a_1c_2 \,||\, b_1d_2e_3 | + | b_1c_2 \,||\, a_1d_2e_3 | ,$$

i.e. $\omega_1 - \omega_{10} + \omega_9 - \omega_6 = 0.$

Similarly

$$\begin{aligned}
\omega_2 + \omega_{10} + \omega_8 - \omega_5 &= 0,\\
\omega_3 - \omega_{10} + \omega_7 - \omega_4 &= 0,\\
\omega_4 + \omega_9 - \omega_3 - \omega_2 &= 0,\\
\omega_5 - \omega_9 - \omega_7 - \omega_2 &= 0,\\
\omega_6 - \omega_8 + \omega_7 - \omega_1 &= 0,\\
\omega_7 + \omega_6 - \omega_5 + \omega_3 &= 0,\\
\omega_8 - \omega_6 - \omega_4 + \omega_2 &= 0,\\
\omega_9 - \omega_5 + \omega_4 + \omega_1 &= 0,\\
\omega_{10} - \omega_3 + \omega_2 - \omega_1 &= 0,\\
\omega_{10} - \omega_9 + \omega_8 - \omega_7 &= 0,\\
\omega_{10} - \omega_5 + \omega_4 + \omega_6 &= 0,\\
\omega_3 - \omega_2 - \omega_9 + \omega_6 &= 0,\\
\omega_3 + \omega_1 + \omega_8 - \omega_5 &= 0,\\
-\omega_7 + \omega_4 - \omega_2 + \omega_1 &= 0
\end{aligned}$$

(cf. Cayley, *Quart. Journ. of Math.* xv. p. 56). What we have now to do is to select any five of these identities, such that five of the ω's involved in them may each occur *three* times. Such a five we have in

$$\begin{aligned}
\omega_1 &= \omega_{10} - \omega_9 + \omega_6,\\
\omega_2 &= \omega_6 - \omega_9 + \omega_3,\\
\omega_4 &= \omega_3 + \omega_7 - \omega_{10},\\
\omega_5 &= \omega_3 + \omega_6 + \omega_7,\\
\omega_8 &= \omega_7 + \omega_9 - \omega_{10};
\end{aligned}$$

and using these to substitute in the result above obtained, viz.

$$\Delta_5 = \omega_1 DE - \omega_2 CE + \ldots + \omega_{10} AB,$$

we have

$$\begin{aligned}\Delta_5 = &\ \ \omega_8\,(\ \ CD - CE + BE - BD)\\ &+\omega_6\,(\ \ BC + DE - CE - BD)\\ &+\omega_7\,(-AE + BE - BD + AD)\\ &+\omega_9\,(-AC - DE + CE + AD)\\ &+\omega_{10}\,(\ \ AB + DE - BE - AD),\\ = &-\omega_8\,(E-D)(C-B)\\ &+\omega_6\,(E-B)(D-C)\\ &+\omega_7\,(E-D)(B-A)\\ &-\omega_9\,(E-A)(D-C)\\ &+\omega_{10}\,(E-A)(D-B),\end{aligned}$$

which is the analogue sought.

Beechcroft, Bishopton, N.B.
Oct. 1883.

FURTHER THEOREMS ON DEFINITE INTEGRALS CONNECTED WITH TAYLOR'S SERIES.

By *E. B. Elliott, M.A.*

1. THE following short paper is a companion to one printed in the January and February numbers (vol. XII. pp. 144-148).

Take the equality

$$\int_{-\infty}^{\infty}\{F(x+h)-F(x)\}\,dx = \int_{-\infty}^{\infty}\int_0^h F'(x+y)\,dx\,dy$$
$$= h\,\{F(\infty)-F(-\infty)\},$$

which holds when it has meaning, *i.e.* when, in addition to the facts as to the values of the functions F, F' for finite arguments that are necessary to make the integrations intelligible, neither of the limits $F(\infty)$, $F(-\infty)$ is infinite, and which takes the special form

$$\int_{-\infty}^{\infty} F(x+h)\,dx = \int_{-\infty}^{\infty} F(x)\,dx,$$

in cases for which the integral on the right of this is of finite value, since then $F(\infty)$ and $F(-\infty)$ vanish; and in it replace

$F(x)$ by $e^{-rx}f^n(x)$, where $f^n(x)$ denotes the nth derived of a function $f(x)$. What has been said as to $F(x)$, and what follows, make sufficiently clear the requirements subject to the satisfaction of which our $f(x)$ can be chosen at will; viz. (1) the possibility of the choice of such a positive integer n, and such a finite or vanishing constant r, that $e^{-rx}f^n(x)$ have limits which are not infinite both for positive and negative infinite values of x, and (2) the usual limitations that for any finite x Taylor's theorem be applicable to the expansion as far as n terms of $f(x+h)$. Our r and n are the particular r and n corresponding as above to a chosen f, and not arbitraries.

Our substitution gives

$$\int_{-\infty}^{\infty} e^{-r(x+h)}f^n(x+h)\,dx - \int_{-\infty}^{\infty} e^{-rx}f^n(x)\,dx = h\left[e^{-rx}f^n(x)\right]_{-\infty}^{\infty},$$

in which the difference of integrals on the left is only for subsequent convenience written as such, and would with greater propriety be written as the integral of a difference, except in the case when each of the two integrals is finite, for which case the right-hand side vanishes. This will have to be remembered in the conclusions to be drawn.

Now multiply by e^{rh}, getting

$$\int_{-\infty}^{\infty} e^{-rx}f^n(x+h)\,dx - e^{rh}\int_{-\infty}^{\infty} e^{-rx}f^n(x)\,dx = \frac{d}{dr}e^{rh}\left[e^{-rx}f^n(x)\right]_{-\infty}^{\infty},$$

and integrate n times, with respect to h, between limits, lower 0 and upper h. There results

$$\int_{-\infty}^{\infty} e^{-rx}\left\{f(x+h)-f(x)-hf'(x)-\frac{h^2}{2!}f''(x)-\ldots-\frac{h^{n-1}}{(n-1)!}f^{n-1}(x)\right\}dx$$
$$-C_n\int_{-\infty}^{\infty} e^{-rx}f^n(x)\,dx = \frac{dC_n}{dr}\left[e^{-rx}f^n(x)\right]_{-\infty}^{\infty},$$

where $$C_n = \int_0^h\int_0^h\ldots\int_0^h e^{rh}\,dh^n;$$

or, as it may be stated,

$$\int_{-\infty}^{\infty} e^{-rx}\left\{\frac{h^n}{n!}f^n(x+\theta h) - C_n f^n(x)\right\}dx = \frac{dC_n}{dr}\left[e^{-rx}f^n(x)\right]_{-\infty}^{\infty},$$

where $\frac{h^n}{n!}f^n(x+\theta h)$ is the remainder after n terms of the expansion of $f(x+h)$ in powers of h by Taylor's Theorem.

The relation thus found we see, bearing in mind a remark made above, to be merely the condensed expression of two distinct theorems, which we now separate and state as follows.

I. If the function $f(x)$ be such that it is possible so to choose the constant r and the number n as, to make the value of $\int_{-\infty}^{\infty} e^{-rx} f^n(x)\, dx$ finite, then for these values

$$\int_{-\infty}^{\infty} e^{-rx}\left\{f(x+h)-f(x)-hf'(x)-\ldots-\frac{h^{n-1}}{(n-1)!}f^{n-1}(x)\right\}dx$$
$$=C_n\int_{-\infty}^{\infty} e^{-rx} f^n(x)\,dx\ldots(A).$$

II. If an n and an r can be chosen, such that this is not the case, but that still, as x increases indefinitely, both positively and negatively the value of $e^{-rx}f^n(x)$, tends to limits neither of which is infinite, then for that n and r

$$\int_{-\infty}^{\infty} e^{-rx}\left\{f(x+h)-f(x)-hf'(x)-\ldots-\frac{h^{n-1}}{(n-1)!}f^{n-1}(x)\right.$$
$$\left.-C_nf^n(x)\right\}dx=\frac{dC_n}{dr}\left[e^{-rx}f^n(x)\right]_{-\infty}^{\infty}\ldots(B).$$

In particular, let us examine the case of $r=0$. We see that

$$C_n^0=\int_0^h\int_0^h\ldots\int_0^h dh^n=\frac{h^n}{n!},$$

and
$$\frac{d}{d0}C_n^0=\int_0^h\int_0^h\ldots\int_0^h h\,dh^n=\frac{h^{n+1}}{(n+1)!};$$

inserting which, in the two results (A), (B), a single one is produced, the latter giving it with $n+1$ for n. The resulting special theorem may be stated:—

III. If $f(x)$ be such a function of x that its $(n-1)^{\text{th}}$ derived function $f^{n-1}(x)$, and no earlier one, has limits $f^{n-1}(\infty), f^{n-1}(-\infty)$ neither of which is infinite, then

$$\int_{-\infty}^{\infty}\left\{f(x+h)-f(x)-hf'(x)-\ldots-\frac{h^{n-1}}{(n-1)!}f^{n-1}(x)\right\}dx$$
$$=\frac{h^n}{n!}\{f^{n-1}(\infty)-f^{n-1}(-\infty)\},$$

which may be more shortly expressed

$$\int_{-\infty}^{\infty} f^n(x+\theta h)\,dx=f^{n-1}(\infty)-f^{n-1}(-\infty)\ldots.(C),$$

a result expressing merely that in such a case as is considered θ is such a function of x that the mean value of $f^n(x+\theta h)$ for all values of x is the same as that of $f^n(x)$.

2. It may be well to illustrate the above results by one or two examples. Take first $f(x) = \log(1+e^x)$, so that

$$f'(x) = \frac{e^x}{1+e^x},\ f''(x) = \frac{e^x}{(1+e^x)^2},\ \text{\&c.}$$

From this it is easy to see that $n=1$, $r=\frac{1}{2}$ gives by (A)

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x} \log\frac{1+e^{x+h}}{1+e^x}\,dx = \pi\int_0^h e^{\frac{1}{2}h}dh = 2\pi\,(e^{\frac{1}{2}h}-1);$$

i.e.
$$\int_0^{\infty} \log\frac{1+k^2y^2}{1+y^2}\,\frac{dy}{y^2} = \pi\,(k-1)\text{............}(\alpha);$$

again, that $n=2$ gives by (C),

$$\int_{-\infty}^{\infty} \left\{\log\frac{1+e^{x+h}}{1+e^x} - \frac{he^x}{1+e^x}\right\} dx = \frac{h^2}{2}\text{............}(\gamma);$$

and, once more, that $n=2$, $r=-1$ gives by (B)

$$\int_{-\infty}^{\infty} e^x \left\{\log\frac{1+e^{x+h}}{1+e^x} - \frac{he^x}{1+e^x} - \frac{e^x}{(1+e^x)^2}\int_0^h\!\!\int_0^h e^{-h}dh^2\right\} dx$$
$$= \int_0^h\!\!\int_0^h e^{-h}h\,dh^2,$$

i.e.
$$\int_{-\infty}^{\infty} e^x \left\{\log\frac{1+e^{x+h}}{1+e^x} - \frac{he^x}{1+e^x} - (h+e^{-h}-1)\frac{e^x}{(1+e^x)^2}\right\} dx$$
$$= h\,(e^{-h}+1) + 2\,(e^{-h}-1)\text{...}(\beta).$$

Take, as another case,

$$f(x) = \tan^{-1}e^x,\quad f'(x) = \frac{e^x}{1+e^{2x}},\quad f''(x) = \frac{e^x(1-e^{2x})}{(1+e^{2x})^2},\ \text{\&c.}$$

From this with $r=-1$, $n=1$ we get by (β)

$$\int_{-\infty}^{\infty} e^x \left\{\tan^{-1}e^{x+h} - \tan^{-1}e^x + (e^{-h}-1)\frac{e^x}{1+e^{2x}}\right\} dx$$
$$= 1-(h+1)\,e^{-h}\text{...}(\beta');$$

and again, also by (β), with $r=1$, $n=2$

$$\int_{-\infty}^{\infty} e^{-x}\left\{\tan^{-1}e^{x+h}-\tan^{-1}e^{x}-\frac{he^{x}}{1+e^{x}}-(e^{h}-1-h)\frac{e^{x}(1-e^{2x})}{(1+e^{2x})^{2}}\right\}dx$$
$$=2(e^{h}-1)-h(e^{h}+1)\ldots(\beta'').$$

It is easy to multiply such examples, and it will be odd if some of the results thus given are not such as can only be obtained otherwise with greater difficulty.

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ON ARCHIMEDES' THEOREM FOR THE SURFACE OF A CYLINDER.

By Professor *Cayley*.

THE measure of the surface of a cylinder was first obtained by Archimedes in his Treatise on the Sphere and Cylinder (BOOK I., Prop. XIV.), *Œuvres d'Archimède*, par F. Peyrard, 4° Paris, 1807, pp. 26—31; viz. Archimedes showed that the surface of the cylinder was equal to the area of a circle radius a mean proportional between the height and the diameter of the circular base [$S=2\pi ah, =\pi\{\sqrt{(2a.h)}\}^{2}$].

The following is *in effect* his demonstration:

He considers regular polygons (with the same number of sides) inscribed in and circumscribed about a circle; and, as regards the cylinder, the prisms standing on these polygons.

Say for the circular base of the cylinder we have

$S^{\times}$ surface of circumscribed prism,
S „ „ cylinder,
S° „ „ inscribed prism;

and for the circle, radius a mean proportional between the height and the diameter of the circular base,

$B^{\times}$ area of circumscribed polygon,
B „ „ circle,
B° „ „ inscribed polygon,

where the four polygons referred to by $S^{\times}$, S°, $B^{\times}$, B° have all of them the same number of sides.

It is in the preceding propositions (by means of an axiom as to curve lines) shown that

$$S^{\times}>S>S^{\circ},\quad B^{\times}>B>B^{\circ};$$

and it is further shown that

$$S^\times = B^\times, \quad S^\circ = B^\circ.$$

It is moreover shown that by taking the number of sides sufficiently large, the ratio $B^\times : B^\circ$, or say the fraction $B^\times/B^\circ$ (which is greater than 1) may be made less than any given quantity $1 + \epsilon$.

It is then to be shown that $S = B$.

If not, then

either $\quad B < S.$

This being so, it is possible to make

$$B^\times/B^\circ < S/B,$$

that is $\quad S^\times/B^\circ < S/B,$

or $\quad S^\times/S < B^\circ/B,$

which is absurd, since

$$S^\times/S > 1; \quad B^\circ/B < 1;$$

or else $\quad B < S.$

This being so, it is possible to make

$$B^\times/B^\circ < B/S,$$

that is $\quad B^\times/S_\circ < B/S,$

or $\quad B^\times/B < S^\circ/S,$

which is absurd, since

$$B^\times/B > 1; \quad S^\circ/S < 1;$$

and consequently $S = B$, the theorem in question.

I take the opportunity of referring to two theorems by Archimedes, Lemmas, Prop. V. and VI., Peyrard, pp. 429-435, which relate to the contacts of circles. We have in each of them the figure which he calls the Arbelon, viz. if A, C, B are points in this order on the same straight line, then the figure consists of the three semicircles on the diameters AC, CB, and AB respectively, and the Arbelon is the space included between the three semi-circumferences.

In Prop. V., we have also the common tangent at C to the two semicircles AC, CB; this divides the Arbelon into two mixtilinear triangles (each bounded by the common tangent, one of the smaller semicircles, and a portion of the larger semicircle), and inscribing each of these a circle, the theorem is that the two inscribed circles are of equal magnitude.

In Prop. VI., the theorem is that the radii of the smaller semicircles being as 3 : 2, then the radius of the circle inscribed in the Arbelon, is to the diameter of the larger semicircle as 6 to 19. But it is noticed that the demonstration would apply to any other value of the ratio; and, in fact, if the radii of the two smaller circles are as $a : b$, then the radius of the inscribed circle is to the diameter of the larger semicircle as ab to $a^2 + ab + b^2$, which is the general form of the theorem.

ON THE POTENTIAL OF AN ELLIPSOID.

By Professor *E. J. Nanson, M.A.*

THE following investigation will be found shorter than the one in Thomson and Tait's *Natural Philosophy*, vol. I., pt. II., pp. 69-72.

The attraction of an elliptic homocoid at an external point x, y, z is $\frac{mp'}{a'b'c'}$, where m is the mass of the homocoid; a', b', c' are the semi-axes of a confocal through x, y, z; and p' is the central perpendicular on the tangent plane at x, y, z.

This theorem, proved geometrically by Adams, is equivalent to the result in § 524 of Thomson and Tait.

Now let a, b, c be the semi-axes of the homocoid, l, m, n the direction cosines of p', and λ the parameter of the confocal through x, y, z, so that

$$a'^2 = a^2 + \lambda, \quad b'^2 = b^2 + \lambda, \quad c'^2 = c^2 + \lambda.$$

Then $$p'^2 = a^2l^2 + b^2m^2 + c^2n^2 + \lambda,$$

therefore $$2p'dp' = d\lambda.$$

Hence, if V be the potential of the homocoid at x, y, z,

$$\frac{dV}{d\lambda} = \frac{1}{2p'}\frac{dV}{dp'} = -\frac{m}{2a'b'c'},$$

and therefore, V being a function of λ only,

$$V = \tfrac{1}{2}m\int_\lambda^\infty \frac{d\lambda}{\sqrt{(a^2+\lambda . b^2+\lambda . c^2+\lambda)}}.$$

This result may also be found from the theorems of Newton and Chasles relating to homocoids, by first calculating the value of the potential of an elliptic homocoid at its centre.

Next, to find the potential at an external point of a heterogeneous ellipsoid, the surfaces of equal density being concentric, similar, and similarly situated ellipsoids, let a, b, c be the semi-axes of the ellipsoid; $a\theta$, $a(\theta+\delta\theta)$ the x semi-axes of an interior homocoid of mass $4\pi\rho abc\theta^2\delta\theta$. The potential of this homocoid at the point x, y, z is

$$2\pi\rho abc\theta^2\delta\theta\int_{\lambda\theta^2}^\infty \frac{dt}{\sqrt{(a^2\theta^2+t . b^2\theta^2+t . c^2\theta^2+t)}},$$

where λ is the positive root of the equation

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=\theta^2.$$

Replacing t by $\theta^2 q$, we find for the potential of the solid ellipsoid

$$V=2\pi abc\int_0^1 \rho\theta\, d\theta\int_\lambda^\infty \frac{dq}{Q},$$

where $$Q=\sqrt{(a^2+q.b^2+q.c^2+q)}.$$

Change the order of integration by the usual process, and we find

$$V=2\pi abc\int_\alpha^\infty \frac{dq}{Q}\int_\beta^1 \rho\theta\, d\theta,$$

where α is the positive root of the equation

$$\frac{x^2}{a^2+\alpha}+\frac{y^2}{b^2+\alpha}+\frac{z^2}{c^2+\alpha}=1.$$

and $$\beta^2=\frac{x^2}{a^2+q}+\frac{y^2}{b^2+q}+\frac{z^2}{c^2+q}.$$

For an internal point we have

$$V=2\pi abc\left\{\int_0^\mu \rho\theta\, d\theta\int_\lambda^\infty \frac{dq}{Q}+\int_\mu^1 \rho\theta\, d\theta\int_0^\infty \frac{dq}{Q}\right\},$$

where $$\mu^2=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}.$$

Changing the order of integration as before, we get

$$V=2\pi abc\int_0^\infty \frac{dq}{Q}\int_\beta^1 \rho\theta\, d\theta.$$

If ρ be constant, we fall upon the usual formula

$$V=2\pi abc\int_\alpha^\infty \frac{dq}{Q}\left(1-\frac{x^2}{a^2+q}-\frac{y^2}{b^2+q}-\frac{z^2}{c^2+q}\right)$$

for the potential of a homogeneous ellipsoid at an external point.

This formula was in effect given by Plana in 1840 (*Crelle*, vol. XX., pp. 271, 272) and is not therefore due to Dirichlet as stated in Thomson and Tait, p. 47. It is of interest to note that Rodrigues in 1815 narrowly missed obtaining the formula for the potential of a homogeneous ellipsoid at an internal point. See Todhunter's *History of Attraction*, Art. 1184.

ON THE GAMMA FUNCTIONS OF A COMPLEX.

By *H. J. Sharpe, M.A.*

In connexion with some investigations on the differential equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + y(x + A) = 0,$$

I have been led to the definite integrals

$$\int_0^\infty \begin{matrix}\cos\\ \sin\end{matrix}(2x^2 + A\log x)\,dx.$$

I am indebted to Professor Stokes for the remarkable Theorem that if we call these two integrals K and S respectively,

$$K^2 + S^2 = \frac{\frac{1}{4}\pi}{e^{\pi A} + 1},$$

He shows that they depend upon the Gamma Functions of a complex. The following is his proof:

Let $$u = \int_0^\infty e^{-x^2 + iA\log x} \times dx \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where $$i = \sqrt{(-1)}.$$

Write ax for x. Then

$$u = ae^{iA\log a}\int_0^\infty e^{-a^2x^2 + iA\log x} \times dx.$$

For a write $$a(\cos\alpha + i\sin\alpha) = ae^{i\alpha}$$

$$u = ae^{i\alpha} \times e^{iA\log a - A\alpha} \times \int_0^\infty e^{-a^2\cos 2\alpha . x^2 - ia^2\sin 2\alpha . x^2 + iA\log x} \times dx.$$

Let $\alpha = -\frac{1}{4}\pi$, then

$$u = ae^{\frac{1}{4}\pi A} . e^{i(A\log a - \frac{1}{4}\pi)} \times \int_0^\infty e^{i(a^2x^2 + A\log x)} \times dx$$

$$= p + iq \text{ suppose} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

For x write $x^{\frac{1}{2}}$ in (1), then

$$u = \tfrac{1}{2}\int_0^\infty e^{-x} x^{\frac{1}{2}(iA)} . x^{-\frac{1}{2}} dx = \tfrac{1}{2}\Gamma\{\tfrac{1}{2} + \tfrac{1}{2}(iA)\};$$

therefore

$$p^2 + q^2 = \tfrac{1}{4}\Gamma\{\tfrac{1}{2} + \tfrac{1}{2}(iA)\}\,\Gamma\{\tfrac{1}{2} - \tfrac{1}{2}(iA)\} = \frac{\pi}{2\{e^{\frac{1}{2}(\pi A)} + e^{-\frac{1}{2}(\pi A)}\}} \ldots(3).$$

Put, for shortness, in (2)

$$A\log a - \tfrac{1}{4}\pi = B, \quad \text{and} \quad \int_0^\infty \begin{matrix}\cos\\ \sin\end{matrix}(a^2x^2 + A\log x)\,dx = \begin{matrix}K\\ S\end{matrix}.$$

Then from (2)

$$(K + iS)\,ae^{\frac{1}{4}(\pi A)} = p\cos B + q\sin B + i(q\cos B - p\sin B),$$

equating real and imaginary parts

$$p\cos B + q\sin B = ae^{\frac{1}{4}(\pi A)}K,$$

$$q\cos B - p\sin B = ae^{\frac{1}{4}(\pi A)}S;$$

therefore, squaring and adding, and using (3),

$$a^2e^{\frac{1}{2}(\pi A)}(K^2 + S^2) = \frac{\pi}{2\left\{e^{\frac{1}{2}(\pi A)} + e^{-\frac{1}{2}(\pi A)}\right\}}.$$

Put $a = \sqrt{2}$, and the Theorem in question follows.

We may notice that at step (3) it is assumed that the formula

$$\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin m\pi}$$

holds when m is unreal. We may establish this point thus. Suppose $m = \frac{1}{2} + \frac{1}{2}iA$. Then we may readily show that

$$\Gamma(m)\Gamma(1-m) = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}x^{iA}y^{-iA}\,dx\,dy$$

change this to polar coordinates, and it

$$= 4\int_0^\infty\int_0^{\frac{1}{2}\pi} e^{-r^2}(\cot\theta)^{iA}\,r\,dr\,d\theta$$

$$= 2\int_0^{\frac{1}{2}\pi}\{\cos(A\log\cot\theta) + i\sin(A\log\cot\theta)\}\,d\theta.$$

The second integral obviously vanishes, and by putting $\cot\theta = e^\phi$, the first is seen to be

$$= 4\int_0^\infty \frac{\cos A\phi\,d\phi}{e^\phi + e^{-\phi}} = \frac{2\pi}{e^{\frac{1}{2}(\pi A)} + e^{-\frac{1}{2}(\pi A)}}$$

by p. 669 of De Morgan's *Differential and Integral Calculus*, or by Gregory's *Examples*, p. 496.

ON THE CONDITIONS OF EQUILIBRIUM OF FORCES IN THREE DIMENSIONS.

By Professor *E. J. Nanson, M.A.*

This paper contains an elementary geometrical investigation of the conditions of equilibrium of forces in three dimensions. The ordinary rules for the composition of forces acting at a point and of couples are assumed, but for the sake of completeness I commence with the following proposition:

Prop. I. Any system of forces can be reduced to a couple G and a single force R acting at an arbitrary point O.

Let P acting at A be one of the forces of the system. At O apply two forces, each equal and parallel to P, in opposite directions. Thus, instead of P at A we have P at O, and a couple formed by P at A and P at O.

Proceeding in this way, we can replace the given system of forces by the following:

(1) A system of forces at O, which are respectively equal and parallel to the original forces: this system may be combined into a single force R at O.

(2) A system of couples which may be combined into a single couple G.

Prop. II. The single force R is such that the component of R in any direction Ox is equal to the sum of the components of the several forces P in the same direction. Also R is the same, whatever be the position of O.

This proposition needs no demonstration.

Prop. III. The single couple G is such that the component of G about Ox is equal to the sum of the moments of the several forces P about Ox.

For let $O'A'$ (fig. 1) be the shortest distance between Ox and the line of action of P, and let θ be the angle which the normal to the plane $O'A'A$ makes with Ox.

At O' apply two forces P_1, P_2, each equal and parallel to P, in opposite directions; also at O, apply P_3 equal parallel

and opposite to P. Then we have, component of couple PP_2 about Ox

$$= \text{sum of components of couples } PP_1,\ P_2P_1 \text{ about } Ox$$
$$= \text{component of couple } PP_1 \text{ about } Ox$$
$$= P.O'A' \cos\theta$$
$$= \text{moment of } P \text{ about } Ox;$$

and similarly for the other forces of the given system. Hence we have component of G about Ox

$$= \text{sum of components of the several couples } PP_2 \text{ about } Ox$$
$$= \text{sum of the moments of the several forces } P \text{ about } Ox.$$

Prop. IV. The necessary and sufficient conditions of equilibrium of a system of forces are that each of the six moments of the system about the edges of a finite tetrahedron should vanish.

For denoting the tetrahedron by $ABCD$, let the system be reduced to a couple and a single force at A. The couple must vanish because the moments of the system about AB, AC, AD are zero. Hence the system, if not in equilibrium, must be reducible to a single force through A. Similarly, it must be reducible to a single force through B, and also to a single force through C, which is absurd.

Prop. V. The necessary and sufficient conditions of equilibrium are that each of the moments of the system about five of the edges of a finite tetrahedron should vanish, and that the resolved part of the system parallel to the edge opposite the sixth edge should vanish.

Let the moments about the edges DA, DB, DC, AB, AC each vanish, and the resolved part of the system parallel to DA vanish. Let the system be reduced to a couple and a single force at D. The couple must vanish because the moments of the system about DA, DB, DC are zero. Hence the system, if not in equilibrium, must be reducible to a single force through D. In the same way it must be reducible to a single force through A. Hence it must be reducible to a single force along DA; but this is impossible by the sixth condition.

Prop. VI. If $ABCD$ be a finite tetrahedron, the necessary and sufficient conditions are that the moments about DA, DB,

DC, BC each vanish, and that the resolved parts parallel to DB, DC vanish.

Since the moments about DA, DB, DC are each zero, the system, if not in equilibrium, reduces to a single force through D. Because the moment about BC is zero, this single force must be in the plane DBC, which is impossible, because the resolved parts parallel to DB, DC are zero.

PROP. VII. The necessary and sufficient conditions are that the moments about AB, AC, DB, DC each vanish, and that the resolved parts parallel to BC and perpendicular to the plane ABC vanish.

Let the system be reduced to a couple and a single force at B. Because the moments about BA, BD are zero the couple must be in the plane ABD, and because the resolved parts parallel to BC and perpendicular to ABC are zero the single force must be perpendicular to BC in the plane ABC. Thus the system is reducible to a couple in the plane ABD, and a force in the plane ABC perpendicular to BC. In like manner it is reducible to a couple in the plane ACD, and to the same single force acting at C in the plane ABC perpendicular to BC. But one of these two equivalents reversed must balance the other, hence three couples in their different places DAB, DAC, DBC must balance. This is impossible unless each couple separately vanish. Hence the couple in the plane ABD and the single force at B must each vanish, and therefore the given system must be in equilibrium.

This proposition may be enunciated as follows:

If the moments of a system about each side of a gauche quadrilateral be zero, and also the resolved parts of the system parallel to one of the diagonals, and perpendicular to that diagonal and one of the sides, be zero, the system is in equilibrium.

PROP. VIII. The necessary and sufficient conditions of equilibrium are that the moments of the system about each of three co-terminous edges of a finite tetrahedron should vanish, and that the resolved parts of the system parallel to the same edges should vanish.

Let DA, DB, DC be the three co-terminous edges, and let the system be reduced to a couple and a single force at D. The couple must vanish because the moments of the system about DA, DB, DC are zero, and the single force must vanish because the resolved parts of the system parallel to DA, DB, DC are zero. Hence the system is in equilibrium.

This proposition, which gives the conditions of equilibrium in the ordinary form, is a particular case of the following general proposition.

PROP. IX. The necessary and sufficient conditions of equilibrium are that the moments of the system about three lines not parallel to one plane should vanish, and that the resolved parts of the system in three non-coplanar directions should vanish.

Let the system be reduced to a couple and a single force at any point O. Since the resolved parts of the system in three non-coplanar directions are zero, the single force must vanish. Hence the system, if not in equilibrium, is equivalent to a couple. But the system cannot be equivalent to a couple, for if it were, the moments of the system would vanish only for lines at right angles to the axis of the couple.

PROOFS OF FEUERBACH'S THEOREM.

By *C. Leudesdorf, M.A.*

EXCEPTING the well-known inversion proof of this theorem by Mr. J. P. Taylor, it is difficult to find one which is direct and tolerably short, and depends only on Elementary Geometry or Trigonometry. As some of the results of an attempt to arrive at such a proof, the following may perhaps be of interest.

Let ABC (fig. 4) be a triangle; O, I, N the centres of its circumscribing, inscribed, nine-point circles respectively; P its orthocentre; E the middle point of the base BC; D the foot of the perpendicular from the vertex A on the base; H the middle point of DE; G the point where the inscribed circle touches BC. Let AI produced meet the base in L and the circumscribing circle in F; and let AO produced meet the same circle in K. And let V be the middle point of the arc DE of the nine-point circle. The circle on AL as diameter will pass through D; let Q be its centre.

1. Since FLC, FCA are clearly similar triangles, $FL : FC :: FC : FA$, so that $FL.FA = FC^2 = FI^2$; therefore at once $EL.ED = EG^2$. Again, $VED = VDE =$ half the angle subtended by DE at the middle point of one of the sides $= \frac{1}{2}(C - B) = LAD$; therefore VE is perpendicular to AL.

[The first of these elementary properties is usually assumed, and reference made for the proof to McDowell's *Exercises* § 154, where it is deduced from two previous propositions. The second may also be seen from the fact that V is the middle point of the arc of the nine-point circle cut off by the line joining the middle points of the sides, so that VE bisects externally the angle formed by the lines joining E to these points, lines which are parallel to the sides.]

2. One of Prof. Casey's proofs (*Quarterly Journal*, vol. IV.; *Sequel to Euclid*, p. 103) depends on the property that the circle with centre V and radius VD or VE cuts orthogonally the inscribed and a-escribed circles. But the proof of this property seems rather complicated. It may be proved more simply as follows: since $EL.ED = EG^2$, the tangents from E to the circles (I) and (Q) are equal; and therefore the same is true with regard to the tangents from V, since $EI^2 - VI^2 = EQ^2 - VQ^2$ on account of the perpendicularity of VE, AL. But VD is the tangent from V to (Q), since the angles VDE, LAD are equal; therefore VD is equal to the tangent from V to (I).

Now, since
$$IG^2 = VI^2 - VD^2,$$
add to each side $\quad VN^2 - 2\,VN.IG$;

thus
$$(VN - IG)^2 = VN^2 - 2\,VN.IG + VI^2 - 2\,VN.\,VH = NI^2;$$
which shews that the nine-point and inscribed circles touch.

Or we may proceed as follows: let VG be joined and produced to meet the nine-point circle in X. Then since $VXD = VED = VDG$, the triangles VGD, VDX are similar, and $VG.\,VX = VD^2$; therefore X lies also on the circle (I). And the circles (N) and (I) will touch at X since their tangents at V and G are parallel, and therefore (tangents to a circle making equal angles with their chord of contact) their tangents at X must coincide.

A similar proof will evidently hold for the a-escribed circle, since it touches the base at a point G' such that $EG = EG'$. If VG' be divided in X' so that $VG'.\,VX' = VD^2$, X' will be the point of contact of this circle with the nine-point circle.

3. The foregoing proof may be expressed more briefly, if put in a rather less elementary manner, as follows: since A is one centre of similitude of the inscribed and a-escribed circles, and L, where a common tangent meets the line of centres, is

clearly the other, the circle (Q), described on AL as diameter, is co-axial with them.

And VE is the common radical axis, being perpendicular to the line of centres and passing through a point E whence tangents EG, EG' to two of the circles are equal. Now since VDE, LAD are equal angles, VD touches the circle (Q); therefore the circle centre V and radius VD cuts orthogonally all three circles. The remainder as in (2); or at once by inversion, according to the method used by Prof. Casey.

4. The proof given by Mr. McMichael (*Messenger*, vol XI. p. 77) depends on a property of the triangle which may by a different treatment be made to supply a very simple proof. First, to prove the property: since as in (1),

$$EL.ED = EG^2 = EG(ED - GD),$$

we have $EG.GD = ED.LG = 2EH.LG = 2IG.VH$

by the similar triangles LIG, VEH.

Next let, as before, VG be joined and produced to meet the circle (N) in X, from which drop XM perpendicular to BC; and describe a circle to pass through X and touch BC at G. Since the tangent to this circle at G is parallel to the tangent at V to the nine-point circle, their tangents at X must coincide in direction; *i.e.* the circle touches the nine-point circle at X. Its centre evidently lies on GI; its radius

$$= \frac{XG^2}{2XM} = \frac{XG.GV}{2VH} = \frac{EG.GD}{2VH} = IG.$$

It must therefore be the inscribed circle itself; and it has already been shewn to touch the nine-point circle at X. Similarly, if G' be the point where the a-escribed circle, centre I', touches BC, it may be shewn that $EG'.G'D = 2I'G'.VH$, and that this circle also touches the nine-point circle.

5. The proof in (4) that $EG.GD = 2IG.VH$ is rather shorter than that given by Mr. McDowell (*Quarterly Journal*, vol. V. p. 269) of what is practically the same property. And it expresses that property in a rather more convenient form; thus, following the same method as that which he adopts,

$$\begin{aligned} NI^2 &= GH^2 + (NH - IG)^2 \\ &= GH^2 + EG.GD + NH^2 - 2IG(NH + VH) + IG^2 \\ &= EH^2 + NH^2 - 2IG.NV + IG^2 \\ &= (NE - IG)^2, \end{aligned}$$

which proves Feuerbach's theorem for the inscribed circle. And similarly for the escribed circle.

[With the help of Trigonometry this proof, and that in (4), may be put very shortly; for

$$EG.GD = FI.IA \sin^2 LAD = 2Rr \sin^2 \tfrac{1}{2}(C-B),$$

$$EG'.G'D = 2Rr_1 \sin^2 \tfrac{1}{2}(C-B),$$

$$VH = VE \sin \tfrac{1}{2}(C-B) = R \sin^2 \tfrac{1}{2}(C-B),$$

so that $EG.GD = 2r.VH$, and $EG'.G'D = 2r_1.VH$.]

6. We may notice that of the properties made use of in (1) and (4) each involves the other immediately. For, without assuming any special property of the circles, we have by equating two expressions for NI^2,

$$VI^2 + VN^2 - 2VN(VH+IG) = GH^2 + (NH-IG)^2;$$

therefore

$$VI^2 - IG^2 - 2VN.VH = 2IG.VH + GH^2 + NH^2 - NV^2,$$

whence $\quad VI^2 - IG^2 - VE^2 = 2IG.VH - EG.GD.$

7. A well-known Trigonometrical method of proof is to calculate the values of OI, OP, IP, and thence NI from the fact that N lies midway between O and P (*Nouvelles Annales* for 1842; *Quarterly Journal*, vol. VII. p. 302, &c.) Similar proofs by Geometry, without the use of Trigonometrical expressions, have been given by Mr. W. F. Walker (*Quarterly Journal*, vol. VIII. p. 47) and by Mr. Richardson (*Educational Times* Reprint, vol. XXXIX. p. 100). The only difficulty in these proofs is to find the value of IP; to shew that $OI^2 = R^2 - 2Rr$ and $OP^2 = R^2 - 2AP.PD$ is easy. The following are two tolerably simple proofs that

$$IP^2 = 2r^2 - AP.PD.$$

Since $FL.FA = FI^2$, the square on the tangent from F to the circle (Q) exceeds the square on that from F to the circle (I) by r^2; and by adding KF^2 to each side the same is seen to be true for the tangents from K to the circles. Also, since $EL.ED = EG^2$ the squares on the tangents to the two circles from E are equal. But since E is the middle point of PK the squares on the tangents from P and K to any circle are clearly together equal to double the square on the tangent from E to that circle, together with $2EP^2$; taking each of the circles (I) and (Q) in turn, and subtracting, we see that the square on the tangent from P to (I) exceeds that to (Q) by r^2, that is,

$$IP^2 - r^2 + AP.PD = r^2, \text{ or } IP^2 = 2r^2 - AP.PD.$$

Otherwise: join AO, and draw In perpendicular to it, and Im perpendicular to AD; these will be equal since IA bisects the angle OAP. Also $AP = 2\,OE$.

And since $\qquad FI^2 = FL.FA$;

therefore at once $\qquad (FE+r)^2 = FE(FE+AD)$;

therefore $\qquad 2r.FE + r^2 = FE.AD$..................(1).

Now
$$OI^2 = In^2 + (AO - An)^2$$
$$IP^2 = Im^2 + (Am - AP)^2;$$

therefore subtracting, since $Im = In$ and $Am = An$
$$IP^2 - OI^2 = AP^2 - 2AmAP + 2AmR - R^2;$$
therefore
$$\begin{aligned} IP^2 + AP.PD &= AP.AD - 2Am.AP + 2AmR - R^2 + OI^2 \\ &= 2\,OE.AD - 4(AD-r)OE + 2R(AD-r) - 2Rr \\ &= 2(AD-2r)(R-OE) \\ &= 2(AD-2r)\,FE \\ &= 2r^2 \text{ by (1).} \end{aligned}$$

Pembroke College, Oxford,
Nov. 7th, 1883.

ON THE QUATERNION TREATMENT OF THE LINEAR COMPLEX.

By *Arthur Buchheim*, *B.A.*, late Scholar of New College, Oxford.

In a former paper (t. XII, p. 129) I shewed how quaternions may be applied to Plücker's Geometry of the straight line. The present note is intended to supplement that paper, so that the two together will contain a fairly complete theory of the linear complex and the linear congruence in parabolic space.

1. *The moment of two complexes.* Let (α, β), (α', β') be the two complexes, then their moment is
$$\Lambda = -\frac{(S\alpha\beta' + S\alpha'\beta)}{T\beta T\beta'}.$$
Now let K, K' be the parameters of the complexes, ϕ the angle between their axes, and Δ the shortest distance of their axes.

We have, since the axes are

$$V\beta\left(\rho+\frac{V\alpha\beta}{\beta^2}\right)=0,\qquad V\beta'\left(\rho+\frac{V\alpha'\beta'}{\beta'^2}\right)=0,$$

$$\Delta=\frac{S\beta\beta'\left(\frac{V\alpha'\beta'}{\beta'^2}-\frac{V\alpha\beta}{\beta^2}\right)}{TV\beta\beta'}$$

$$=\frac{S\beta\beta'\,(\beta^2V\alpha'\beta'-\beta'^2V\alpha\beta)}{\beta^2\beta'^2TV\beta\beta'}$$

$$=\frac{S\beta\beta'\,(\beta^2V\alpha'\beta'-\beta'^2V\alpha\beta)}{T^3\beta T^3\beta'\sin\phi}.$$

Therefore

$$T^3\beta T^3\beta'\Delta\sin\phi=\beta^2SV\beta\beta'V\alpha'\beta'-\beta'^2SV\beta\beta'V\alpha\beta.$$

But $\qquad SV\alpha\beta V\gamma\delta=S\alpha\delta S\beta\gamma-S\alpha\gamma S\beta\delta.$

Therefore $\qquad T^3\beta T^3\beta'\gamma'\Delta\sin\phi$

$$=\beta^2(S\beta\beta'S\alpha'\beta'-\beta'^2S\alpha'\beta)-\beta'^2(\beta^2S\alpha\beta'-S\beta'\beta S\alpha\beta).$$

Therefore

$$\Delta\sin\phi=-\frac{(S\alpha\beta'+S\alpha'\beta)}{T\beta T\beta'}+\frac{S\beta\beta'}{T\beta T\beta'}\left(\frac{S\alpha'\beta'}{\beta'^2}+\frac{S\alpha\beta}{\beta^2}\right)$$

$$=\Lambda-(K+K')\cos\phi.$$

That is, $\qquad \Lambda=\Delta\sin\phi+(K+K')\cos\phi.$

Therefore the conditions that the axes of two complexes may intersect at right angles are $\Lambda=0$, $\cos\phi=0$, that is, $S\alpha\beta'+S\alpha'\beta=0$, $S\beta\beta'=0$.

2. Every congruence contains one, and in general only one pair of complexes whose axes intersect at right angles.

For if the axes of $(\alpha+\mu\alpha',\ \beta+\mu\beta')$, $(\alpha+\mu'\alpha',\ \beta+\mu'\beta')$ are to intersect at right angles, we must have by what has just been proved

$$S(\alpha+\mu\alpha')(\beta+\mu'\beta')+S(\alpha+\mu'\alpha')(\beta+\mu\beta')=0,$$

and $\qquad S(\beta+\mu\beta')(\beta+\mu'\beta')=0,$

that is $\quad 2S\alpha\beta+(\mu+\mu')(S\alpha\beta'+S\alpha'\beta)+2\mu\mu'S\alpha'\beta'=0,$

$$\beta^2+(\mu+\mu')\,S\beta\beta'+\mu\mu'\,\beta'^2=0.$$

Therefore μ, μ' are the roots of the quadratic

$$\begin{vmatrix} \mu^2, & -\mu, & 1 \\ 2S\alpha\beta, & S\alpha\beta' + S\alpha'\beta, & 2S\alpha'\beta' \\ \beta^2, & S\beta\beta', & \beta'^2 \end{vmatrix} = 0,$$

or, taking as we may $T\beta = T\beta' = 1$, the quadratic is

$$\begin{vmatrix} \mu^2, & -\mu, & 1 \\ 2K, & \Lambda, & 2K' \\ 1, & \cos\phi, & 1 \end{vmatrix} = 0.$$

3. If the axes of two complexes intersect, they can be simultaneously deduced to the canonical form.

For if a complex is to be in the canonical form, the origin must be on its axis; therefore two complexes can be simultaneously reduced to the canonical form, if and only if their axes intersect; the origin will then be the point of intersection of the axes.

4. Taking the two complexes obtained in (2), we see by (3) that they may both be taken in the canonical form; so that if K, K' are their parameters, the complexes are $(K\alpha, \alpha)$, $(K'\beta, \beta)$ with the condition $S\alpha\beta = 0$; we can also suppose that $T\alpha = T\beta = 1$.

Then any complex of the congruence will have for its coordinates $K\alpha + \mu K'\beta$, $\alpha + \mu\beta$, and if ρ is any point on its axis, we have, ξ being a scalar,

$$\rho = -\frac{V(K\alpha + \mu K'\beta)(\alpha + \mu\beta)}{(\alpha + \mu\beta)^2} + \xi(\alpha + \mu\beta)$$

$$= \frac{(K - K')\mu V\alpha\beta}{1 + \mu^2} + \xi(\alpha + \mu\beta).$$

Therefore

$$S\alpha\rho = -\xi, \quad S\beta\rho = -\xi\mu, \quad S\rho\alpha\beta = -\frac{(K - K')\mu}{1 + \mu^2};$$

therefore

$$S\rho\alpha\beta(S^2\alpha\rho + S^2\beta\rho) = -(K - K')\zeta^2\mu = -(K - K')S\alpha\rho S\beta\rho;$$

therefore $S\rho\alpha\beta(S^2\alpha\rho + S^2\beta\rho) + (K - K')S\alpha\rho S\beta\rho = 0.$

That is, every point on the axis of any complex of the congruence satisfies this equation; therefore taking α, β, $V\alpha\beta$ as axes,

$$z(x^2 + y^2) - (K - K')xy = 0,$$

is the equation to the surface generated by the axes of the complexes of the congruence; viz. it is the equation to the cylindroid.

5. The intercept on $V\alpha\beta$, made by the axis of a complex, is

$$z=(K-K')\frac{\mu}{1+\mu^2}.$$

The angle between α and the axis of the complex is given by

$$\cos\phi=\frac{1}{\sqrt{(1+\mu^2)}},$$

so that $$\tan\phi=\mu.$$

The parameter of the complex is

$$K''=\frac{S(\alpha+\mu\rho)(K\alpha+\mu K'\beta)}{(\alpha+\mu\beta)^2}$$
$$=\frac{K+\mu^2K'}{1+\mu^2},$$

we therefore have $$z=\tfrac{1}{2}(K-K')\sin 2\theta,$$
$$K''=K\cos^2\phi+K'\sin^2\phi.$$

6. For a special complex we have $K''=0$; therefore $\mu^2=-\frac{K}{K'}$: therefore K, K' must be of opposite signs if the special complexes are to be real: let $K'=-l$. Then

$$M=\pm\left(\frac{K}{l}\right)^{\frac{1}{2}},$$
$$\phi=\tan^{-1}\pm\left(\frac{K}{l}\right)^{\frac{1}{2}},$$
$$z=\pm\sqrt{(Kl)}.$$

7. In connection with (1) above, I add the following investigation of the shortest distance between two lines; its special point is, that it avoids the use of the scalar variable in the equation of a straight line.

Let the lines be

$$V\alpha(\rho-\beta)=0,\quad V\alpha'(\rho-\beta')=0;$$

let ρ, ρ' be the extremities of the shortest distance, we have

$$V\alpha\rho=V\alpha\beta.$$

Operate with $S.\alpha'$, then

$$S\alpha\alpha'\rho = S\alpha\alpha'\beta.$$

Similarly $$S\alpha\alpha'\rho' = S\alpha\alpha'\beta';$$

therefore $$S\alpha\alpha'(\rho-\rho') = S\alpha\alpha'(\beta-\beta').$$

But $V.(\rho-\rho')\,V\alpha\alpha' = 0$, and therefore

$$S\alpha\alpha'(\rho-\rho') = TV\alpha\alpha'\,T(\rho-\rho');$$

therefore $$T(\rho-\rho') = \frac{S\alpha\alpha'(\beta-\beta')}{TV\alpha\alpha'}.$$

DÉMONSTRATION ANALYTIQUE DU THÉORÈME DE M. S. ROBERTS SUR QUATRE SPHÈRES CONCOURANTES.

Par *M. J. Neuberg*, Professeur à l'université de Liège (Belgique).

SI sur chaque arête d'un tétraèdre $ABCD$, on marque arbitrairement un point, les sphères α, β, γ, δ décrites par un sommet du tétraèdre et par les points marqués sur les trois arêtes adjacentes passent par un même point. (*Proceedings of the London Mathematical Society*, vol. XII., No. 173).

Prenons $ABCD$ pour tétraèdre de référence, et soient

$$M \equiv M_{12}\lambda_1\lambda_2 + M_{13}\lambda_1\lambda_3 + M_{14}\lambda_1\lambda_4 + M_{23}\lambda_2\lambda_3 + M_{24}\lambda_2\lambda_4 + M_{34}\lambda_3\lambda_4 = 0,$$

$$N \equiv M_1\lambda_1 + N_2\lambda_2 + N_3\lambda_3 + N_4\lambda_4 = 0,$$

les équations de la sphère $ABCD$ et du plan de l'infini. Les sphères α, β, γ, δ ont des équations de la forme

$$M + NP_1 = 0,\quad M + NP_2 = 0,\quad M + NP_3 = 0,\quad M + NP_2 = 0,$$

dans lesquelles

$$P_1 \equiv P_{12}\lambda_2 + P_{13}\lambda_3 + P_{14}\lambda_4,$$
$$P_2 \equiv P_{21}\lambda_1 + P_{23}\lambda_3 + P_{24}\lambda_4,$$
$$P_3 \equiv P_{31}\lambda_1 + P_{32}\lambda_2 + P_{34}\lambda_4,$$
$$P_4 \equiv P_{41}\lambda_1 + P_{42}\lambda_2 + P_{43}\lambda_3.$$

Exprimons maintenant que les sphères α, β coupent AB au même point, si dans leurs équations on fait $\lambda_3 = 0$, $\lambda_4 = 0$, on trouve

$$M_{12}\lambda_1\lambda_2 + (N_1\lambda_1 + N_2\lambda_2)\, P_{12}\lambda_2 = 0,$$

$$M_{12}\lambda_1\lambda_2 + (N_1\lambda_1 + N_2\lambda_2)\, P_{21}\lambda_1 = 0\,;$$

d'où, en divisant ces égalités respectivement par λ_2 et λ_1, et en égalant ensuite les deux valeurs du rapport $\lambda_1 : \lambda_2$,

$$M_{12} = -N_1P_{12} - N_2P_{21}.$$

Si l'on opère de même sur les autres arêtes de $ABCD$, on trouve

$$-M \equiv \Sigma\,(N_1P_{12}+N_2P_{21})\,\lambda_1\lambda_2 \equiv N_1P_1\lambda_1 + N_2P_2\lambda_2 + N_3P_3\lambda_3 + N_4P_4\lambda_4.$$

L'équation de α est donc

$$N_1P_1\lambda_1 + N_2P_2\lambda_2 + N_3P_3\lambda_3 + N_4P_4\lambda_4 - NP_1 = 0 \ldots\ldots (1).$$

Le centre radical des sphères α, β, γ, δ satisfait aux équations

$$P_1 = P_2 = P_3 = P_4\,;$$

mais il vérifie aussi l'équation (1); car celle-ci, si l'on remplace P_2, P_3, P_4 par P_1, se réduit à l'identité

$$P_1\,(N_1\lambda_1 + N_2\lambda_2 + N_3\lambda_3 + N_4\lambda_4) - NP_1 = 0.$$

Donc les quatre sphères α, β, γ, δ passent par un même point.

La démonstration précédente fait voir immédiatement qu'on peut remplacer les sphères α, β, γ, δ par quatre surfaces du second dégré ayant une conique en commun.

[Mr. Roberts remarks that M. Neuberg's very symmetrical process is equally applicable to the corresponding functions of more than four homogeneous variables, so that we have a complete series of similar theorems for hyperspaces].

NOTE ON SOME ALGEBRAICAL FORMULÆ CONNECTED WITH A SYSTEM OF q-SERIES IN ELLIPTIC FUNCTIONS.

By *J. W. L. Glaisher.*

§ 1. Of all the numerous and remarkable series of algebraical theorems and identities which are deducible at once from formulæ in the Theory of Elliptic Functions, the system which is derived from the q-series for $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ and their products seem to be one of the most complete and curious.

Putting $$\rho = \frac{2K}{\pi}, \quad u = \frac{2Kx}{\pi},$$

this system of q-series may be written

$$k\rho \operatorname{sn} u = \sum_1^\infty \frac{4q^{\frac{1}{2}(2n-1)}}{1-q^{2n-1}} \sin(2n-1)x,$$

$$k\rho \operatorname{cn} u = \sum_1^\infty \frac{4q^{\frac{1}{2}(2n-1)}}{1+q^{2n-1}} \cos(2n-1)x,$$

$$\rho \operatorname{dn} u = 1 + \sum_1^\infty \frac{4q^n}{1+q^{2n}} \cos 2nx,$$

$$k\rho^2 \operatorname{cn} u \operatorname{dn} u = \sum_1^\infty \frac{(8n-4)\,q^{\frac{1}{2}(2n-1)}}{1-q^{2n-1}} \cos(2n-1)x,$$

$$k\rho^2 \operatorname{sn} u \operatorname{dn} u = \sum_1^\infty \frac{(8n-4)\,q^{\frac{1}{2}(2n-1)}}{1+q^{2n-1}} \sin(2n-1)x,$$

$$k^2\rho^2 \operatorname{sn} u \operatorname{cn} u = \sum_1^\infty \frac{8nq^n}{1+q^{2n}} \sin 2nx,$$

$$k^2\rho^3 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u = \sum_1^\infty \frac{8n^2q^n}{1-q^{2n}} \sin 2nx.$$

§ 2. Putting in these formulæ, $q = e^{-2a}$, it follows therefore that, if

$$A = \quad 2\sum_{1}^{\infty} \frac{\sin(2n-1)x}{\sinh(2n-1)a} = \sum_{-\infty}^{\infty} \frac{\sin(2n-1)x}{\sinh(2n-1)a},$$

$$B = \quad 2\sum_{1}^{\infty} \frac{\cos(2n-1)x}{\cosh(2n-1)a} = \sum_{-\infty}^{\infty} \frac{\cos(2n-1)x}{\cosh(2n-1)a},$$

$$C = 1 + 2\sum_{1}^{\infty} \frac{\cos 2nx}{\cosh 2na} = \sum_{-\infty}^{\infty} \frac{\cos 2nx}{\cosh 2na},$$

then

$$AB = 2\sum_{1}^{\infty} (2n) \frac{\sin 2nx}{\cosh 2na} = \sum_{-\infty}^{\infty} (2n) \frac{\sin 2nx}{\cosh 2na},$$

$$AC = 2\sum_{1}^{\infty} (2n-1) \frac{\sin(2n-1)x}{\cosh(2n-1)a} = \sum_{-\infty}^{\infty} (2n-1) \frac{\sin(2n-1)x}{\cosh(2n-1)a},$$

$$BC = 2\sum_{1}^{\infty} (2n-1) \frac{\cos(2n-1)x}{\sinh(2n-1)a} = \sum_{-\infty}^{\infty} (2n-1) \frac{\cos(2n-1)x}{\sinh(2n-1)a},$$

and

$$ABC = \sum_{1}^{\infty} (2n)^2 \frac{\sin 2nx}{\sinh 2na} = \tfrac{1}{2}\sum_{-\infty}^{\infty} (2n)^2 \frac{\sin 2nx}{\sinh 2na}.$$

In these equations a may denote any quantity real or complex, except a pure imaginary; *i. e.* we have $a = \alpha + i\beta$, where α and β are unrestricted except that α must not be zero. If x be not real, but $= \gamma + i\delta$, then it is further necessary for the convergence of the series that δ should be numerically less than α.

It will be noticed that by adding the second and third equations, and the fourth and fifth, we have also

$$B + C = 1 + 2\sum_{1}^{\infty} \frac{\cos nx}{\cosh na} = \sum_{-\infty}^{\infty} \frac{\cos nx}{\cosh na},$$

and

$$A(B + C) = \quad 2\sum_{1}^{\infty} n \frac{\sin nx}{\cosh na} = \sum_{-\infty}^{\infty} n \frac{\sin nx}{\cosh na}.$$

§ 3. Writing, for convenience,

$$n_1 = 2n - 1, \quad \text{and} \quad n_2 = 2n,$$

so that n_1 denotes any uneven nnmber and n_2 any even number, we have therefore, identically,

$$\sum_{-\infty}^{\infty} \frac{\sin n_1 x}{\sinh n_1 a} \times \sum_{-\infty}^{\infty} \frac{\cos n_1 x}{\cosh n_1 a} = \sum_{-\infty}^{\infty} n_2 \frac{\sin n_2 x}{\cosh n_2 a},$$

$$\sum_{-\infty}^{\infty} \frac{\sin n_1 x}{\sinh n_1 a} \times \sum_{-\infty}^{\infty} \frac{\cos n_2 x}{\cosh n_2 a} = \sum_{-\infty}^{\infty} n_1 \frac{\sin n_1 x}{\cosh n_1 a},$$

$$\sum_{-\infty}^{\infty} \frac{\cos n_1 x}{\cosh n_1 a} \times \sum_{-\infty}^{\infty} \frac{\cos n_2 x}{\cosh n_2 a} = \sum_{-\infty}^{\infty} n_1 \frac{\cos n_1 x}{\sinh n_1 a},$$

$$\begin{aligned} \sum_{-\infty}^{\infty} \frac{\sin n_1 x}{\sinh n_1 a} \times \sum_{-\infty}^{\infty} \frac{\cos n_1 x}{\cosh n_1 a} \times \sum_{-\infty}^{\infty} \frac{\cos n_2 x}{\cosh n_2 a} & \\ = \sum_{-\infty}^{\infty} \frac{\sin n_1 x}{\sinh n_1 a} \times \sum_{-\infty}^{\infty} n_1 \frac{\cos n_1 x}{\sinh n_1 a} & \\ = \sum_{-\infty}^{\infty} \frac{\cos n_1 x}{\cosh n_1 a} \times \sum_{-\infty}^{\infty} n_1 \frac{\sin n_1 x}{\cosh n_1 a} & \\ = \sum_{-\infty}^{\infty} \frac{\cos n_2 x}{\cosh n_2 a} \times \sum_{-\infty}^{\infty} n_2 \frac{\sin n_2 x}{\cosh n_2 a} & \\ = \tfrac{1}{2} \sum_{-\infty}^{\infty} n_2^2 \frac{\sin n_2 x}{\sinh n_2 a}. & \end{aligned}$$

Also, by addition from the first two equations, we find

$$\sum_{-\infty}^{\infty} \frac{\sin n_1 x}{\sinh n_1 a} \times \sum_{-\infty}^{\infty} \frac{\cos nx}{\cosh na} = \sum_{-\infty}^{\infty} n \frac{\sin nx}{\cosh na}.$$

§ 4. By making x very small in these equations we deduce the following system of formulæ involving a only:

$$\sum_{-\infty}^{\infty} \frac{n_1}{\sinh n_1 a} \times \sum_{-\infty}^{\infty} \frac{1}{\cosh n_1 a} = \sum_{-\infty}^{\infty} \frac{n_2^2}{\cosh n_2 a},$$

$$\sum_{-\infty}^{\infty} \frac{n_1}{\sinh n_1 a} \times \sum_{-\infty}^{\infty} \frac{1}{\cosh n_2 a} = \sum_{-\infty}^{\infty} \frac{n_1^2}{\cosh n_1 a},$$

$$\sum_{-\infty}^{\infty} \frac{1}{\cosh n_1 a} \times \sum_{-\infty}^{\infty} \frac{1}{\cosh n_2 a} = \sum_{-\infty}^{\infty} \frac{n_1}{\sinh n_1 a},$$

$$\sum_{-\infty}^{\infty}\frac{n_1}{\sinh n_1a}\times\sum_{-\infty}^{\infty}\frac{1}{\cosh n_1a}\times\sum_{-\infty}^{\infty}\frac{1}{\cosh n_2a}$$

$$=\sum_{-\infty}^{\infty}\frac{n_1}{\sinh n_1a}\times\sum_{-\infty}^{\infty}\frac{n_1}{\sinh n_1a}$$

$$=\sum_{-\infty}^{\infty}\frac{1}{\cosh n_1a}\times\sum_{-\infty}^{\infty}\frac{n_1^2}{\cosh n_1a}$$

$$=\sum_{-\infty}^{\infty}\frac{1}{\cosh n_2a}\times\sum_{-\infty}^{\infty}\frac{n_2^2}{\cosh n_2a}$$

$$=\tfrac{1}{2}\sum_{-\infty}^{\infty}\frac{n_2^3}{\sinh n_2a};$$

and also

$$\sum_{-\infty}^{\infty}\frac{n_1}{\sinh n_1a}\times\sum_{-\infty}^{\infty}\frac{1}{\cosh na}=\sum_{-\infty}^{\infty}\frac{n^2}{\cosh na}.$$

§ 5. Similarly, by putting $x=\frac{1}{2}\pi-\varepsilon$ where ε is very small, so that

$$\sin n_1x=(-1)^{\frac{1}{2}(n_1-1)}\quad,\quad \cos n_1x=(-1)^{\frac{1}{2}(n_1-1)}n_1\varepsilon,$$

$$\sin n_2x=(-1)^{\frac{1}{2}(n_2-2)}n_2\varepsilon,\quad \cos n_2x=(-1)^{\frac{1}{2}n_2},$$

we obtain the formulæ:

$$\sum_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(n_1-1)}}{\sinh n_1a}\times\sum_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(n_1-1)}n_1}{\cosh n_1a}=\sum_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(n_2-2)}n_2^2}{\cosh n_2a},$$

$$\sum_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(n_1-1)}}{\sinh n_1a}\times\sum_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}n_2}}{\cosh n_2a}=\sum_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(n_1-1)}n_1}{\cosh n_1a},$$

$$\sum_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(n_1-1)}n_1}{\cosh n_1a}\times\sum_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}n_2}}{\cosh n_2a}=\sum_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(n_1-1)}n_1^2}{\sinh n_1a},$$

$$\sum_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(n_1-1)}}{\sinh n_1a}\times\sum_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(n_1-1)}n_1}{\cosh n_1a}\times\sum_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}n_2}}{\cosh n_2a}$$

$$=\sum_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(n_1-1)}}{\sinh n_1a}\times\sum_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(n_1-1)}n_1^2}{\sinh n_1a}$$

$$=\sum_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(n_1-1)}n_1}{\cosh n_1a}\times\sum_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(n_1-1)}n_1}{\cosh n_1a}$$

$$= \sum_{-\infty}^{\infty} \frac{(-1)^{\frac{1}{2}n_2}}{\cosh n_2 a} \times \sum_{-\infty}^{\infty} \frac{(-1)^{\frac{1}{2}(n_2-2)} n_2^{\,2}}{\cosh n_2 a}$$

$$= \tfrac{1}{2} \sum_{-\infty}^{\infty} \frac{(-1)^{\frac{1}{2}(n_2-2)} n_2^{\,3}}{\sinh n_2 a}.$$

In all the preceding formulæ, as written, the real part of a must be positive; if it be negative, all expressions involving the function sinh require to have the negative sign prefixed to them.*

§6. The formulæ contained in the two preceding sections are purely algebraical. Expressing them, therefore, algebraically, instead of by the aid of the functions sinh and cosh, we obtain the following formulæ, in which x denotes any quantity real or complex, whose analytical modulus is less than unity.

From §4 we have

$$\sum_{-\infty}^{\infty} \frac{(2n-1)x^{2n-1}}{1-x^{4n-2}} \times \sum_{-\infty}^{\infty} \frac{x^{2n-1}}{1+x^{4n-2}} = \tfrac{1}{2} \sum_{-\infty}^{\infty} \frac{(2n)^2 x^{2n}}{1+x^{4n}},$$

$$\sum_{-\infty}^{\infty} \frac{(2n-1)x^{2n-1}}{1-x^{4n-2}} \times \sum_{-\infty}^{\infty} \frac{x^{2n}}{1+x^{4n}} = \tfrac{1}{2} \sum_{-\infty}^{\infty} \frac{(2n-1)^2 x^{2n-1}}{1+x^{4n-2}},$$

$$\sum_{-\infty}^{\infty} \frac{x^{2n-1}}{1+x^{4n-2}} \times \sum_{-\infty}^{\infty} \frac{x^{2n}}{1+x^{4n}} = \tfrac{1}{2} \sum_{-\infty}^{\infty} \frac{(2n-1)x^{2n-1}}{1-x^{4n-2}},$$

$$\sum_{-\infty}^{\infty} \frac{(2n-1)x^{2n-1}}{1-x^{4n-2}} \times \sum_{-\infty}^{\infty} \frac{x^{2n-1}}{1+x^{4n-2}} \times \sum_{-\infty}^{\infty} \frac{x^{2n}}{1+x^{4n}}$$

$$= \tfrac{1}{2} \sum_{-\infty}^{\infty} \frac{(2n-1)x^{2n-1}}{1-x^{4n-2}} \times \sum_{-\infty}^{\infty} \frac{(2n-1)x^{2n-1}}{1-x^{4n-2}}$$

$$= \tfrac{1}{2} \sum_{-\infty}^{\infty} \frac{x^{2n-1}}{1+x^{4n-2}} \times \sum_{-\infty}^{\infty} \frac{(2n-1)^2 x^{2n-1}}{1+x^{4n-2}}$$

$$= \tfrac{1}{2} \sum_{-\infty}^{\infty} \frac{x^{2n}}{1+x^{4n}} \times \sum_{-\infty}^{\infty} \frac{(2n)^2 x^{2n}}{1+x^{4n}}$$

$$= \tfrac{1}{8} \sum_{-\infty}^{\infty} \frac{(2n)^3 x^{2n}}{1-x^{4n}},$$

* It should have been stated on p. 127 that a must be positive, otherwise the negative sign should be prefixed to the expressions for A, BC, and ABC.

and

$$\sum_{-\infty}^{\infty} \frac{(2n-1)\,x^{2n-1}}{1-x^{4n-2}} \times \sum_{-\infty}^{\infty} \frac{x^n}{1+x^{2n}} = \tfrac{1}{2} \sum_{-\infty}^{\infty} \frac{n^2 x^n}{1+x^{2n}}.$$

From § 5 we have

$$\sum_{-\infty}^{\infty} (-)^{n-1} \frac{x^{2n-1}}{1-x^{4n-2}} \times \sum_{-\infty}^{\infty} (-)^{n-1} \frac{(2n-1)\,x^{2n-1}}{1+x^{4n-2}}$$
$$= \tfrac{1}{2} \sum_{-\infty}^{\infty} (-)^{n-1} \frac{(2n)^2\, x^{2n}}{1+x^{4n}},$$

$$\sum_{-\infty}^{\infty} (-)^{n-1} \frac{x^{2n-1}}{1-x^{4n-2}} \times \sum_{-\infty}^{\infty} (-)^{n} \frac{x^{2n}}{1+x^{4n}}$$
$$= \tfrac{1}{2} \sum_{-\infty}^{\infty} (-)^{n-1} \frac{(2n-1)\,x^{2n-1}}{1+x^{4n-2}},$$

$$\sum_{-\infty}^{\infty} (-)^{n-1} \frac{(2n-1)\,x^{2n-1}}{1+x^{4n-2}} \times \sum_{-\infty}^{\infty} (-)^{n} \frac{x^{2n}}{1+x^{4n}}$$
$$= \tfrac{1}{2} \sum_{-\infty}^{\infty} (-)^{n-1} \frac{(2n-1)^2\, x^{2n-1}}{1-x^{4n-2}},$$

$$\sum_{-\infty}^{\infty} (-)^{n-1} \frac{x^{2n-1}}{1-x^{4n-2}} \times \sum_{-\infty}^{\infty} (-)^{n-1} \frac{(2n-1)\,x^{2n-1}}{1+x^{4n-2}}$$
$$\times \sum_{-\infty}^{\infty} (-)^{n} \frac{x^{2n}}{1+x^{4n}}$$
$$= \tfrac{1}{2} \sum_{-\infty}^{\infty} (-)^{n-1} \frac{x^{2n-1}}{1-x^{4n-2}} \times \sum_{-\infty}^{\infty} (-)^{n-1} \frac{(2n-1)^2\, x^{2n-1}}{1-x^{4n-2}}$$
$$= \tfrac{1}{2} \sum_{-\infty}^{\infty} (-)^{n-1} \frac{(2n-1)\,x^{2n-1}}{1+x^{4n-2}} \times \sum_{-\infty}^{\infty} (-)^{n-1} \frac{(2n-1)\,x^{2n-1}}{1+x^{4n-2}}$$
$$= \tfrac{1}{2} \sum_{-\infty}^{\infty} (-)^{n} \frac{x^{2n}}{1+x^{4n}} \times \sum_{-\infty}^{\infty} (-)^{n-1} \frac{(2n)^2\, x^{2n}}{1+x^{4n}}$$
$$= \tfrac{1}{8} \sum_{-\infty}^{\infty} (-)^{n-1} \frac{(2n)^3\, x^{2n}}{1-x^{4n}}.$$

By expanding the sines and cosines in the equations in § 3 and equating the coefficients of x^2 or x^3 in the different equations, we obtain the formulæ:

$$\sum_{-\infty}^{\infty}\frac{n_1^3}{\sinh n_1 a}\times\sum_{-\infty}^{\infty}\frac{1}{\cosh n_1 a}+3\sum_{-\infty}^{\infty}\frac{n_1}{\sinh n_1 a}\times\sum_{-\infty}^{\infty}\frac{n_1^2}{\cosh n_1 a}$$
$$=\sum_{-\infty}^{\infty}\frac{n_2^4}{\cosh n_2 a},$$

$$\sum_{-\infty}^{\infty}\frac{n_1^3}{\sinh n_1 a}\times\sum_{-\infty}^{\infty}\frac{1}{\cosh n_2 a}+3\sum_{-\infty}^{\infty}\frac{n_1}{\sinh n_1 a}\times\sum_{-\infty}^{\infty}\frac{n_2^2}{\cosh n_2 a}$$
$$=\sum_{-\infty}^{\infty}\frac{n_1^4}{\cosh n_1 a},$$

$$\sum_{-\infty}^{\infty}\frac{n_1^3}{\sinh n_1 a}\times\sum_{-\infty}^{\infty}\frac{1}{\cosh na}+3\sum_{-\infty}^{\infty}\frac{n_1}{\sinh n_1 a}\times\sum_{-\infty}^{\infty}\frac{n^2}{\cosh n a}$$
$$=\sum_{-\infty}^{\infty}\frac{n^4}{\cosh na},$$

$$\sum_{-\infty}^{\infty}\frac{1}{\cosh n_1 a}\times\sum_{-\infty}^{\infty}\frac{n_2^2}{\cosh n_2 a}+\sum_{-\infty}^{\infty}\frac{1}{\cosh_2 na}\times\sum_{-\infty}^{\infty}\frac{n_1^2}{\cosh n_1 a}$$
$$=\sum_{-\infty}^{\infty}\frac{n_1^3}{\sinh n_1 a},$$

$$\sum_{-\infty}^{\infty}\frac{1}{\cosh n_1 a}\times\sum_{-\infty}^{\infty}\frac{n_1^4}{\cosh n_1 a}+3\sum_{-\infty}^{\infty}\frac{n_1^2}{\cosh n_1 a}\times\sum_{-\infty}^{\infty}\frac{n_1^2}{\cosh n_1 a}$$
$$=\sum_{-\infty}^{\infty}\frac{1}{\cosh n_2 a}\times\sum_{-\infty}^{\infty}\frac{n_2^4}{\cosh n_2 a}+3\sum_{-\infty}^{\infty}\frac{n_2^2}{\cosh n_2 a}\times\sum_{-\infty}^{\infty}\frac{n_2^2}{\cosh n_2 a}$$
$$=4\sum_{-\infty}^{\infty}\frac{n_1}{\sinh n_1 a}\times\sum_{-\infty}^{\infty}\frac{n^3}{\sinh n_1 a}$$
$$=\tfrac{1}{2}\sum_{-\infty}^{\infty}\frac{n_2^5}{\sinh n_2 a}.$$

This last equation may be expressed algebraically in the form

$$\sum_{-\infty}^{\infty}\frac{(2n-1)\,x^{2n-1}}{1-x^{4n-2}}\times\sum_{-\infty}^{\infty}\frac{(2n-1)^3x^{2n-1}}{1-x^{4n-2}}=\tfrac{1}{16}\sum_{-\infty}^{\infty}\frac{(2n)^5x^{2n}}{1-x^{4n}}.$$

§ 6. It may be remarked also that by equating the coefficient of x^5 in the equation

$$\sum_{-\infty}^{\infty} \frac{\sin n_1 x}{\sinh n_1 a} \times \sum_{-\infty}^{\infty} n_1 \frac{\cos n_1 x}{\sinh n_1 a} = \tfrac{1}{2} \sum_{-\infty}^{\infty} n_2^2 \frac{\sin n_2 x}{\sinh n_2 a},$$

we find

$$3\sum_{-\infty}^{\infty} \frac{n_1}{\sinh n_1 a} \times \sum_{-\infty}^{\infty} \frac{n_1^5}{\sinh n_1 a} + 5\sum_{-\infty}^{\infty} \frac{n_1^3}{\sinh n_1 a} \times \sum_{-\infty}^{\infty} \frac{n_1^3}{\sinh n_1 a}$$
$$= \tfrac{1}{4} \sum_{-\infty}^{\infty} \frac{n_2^7}{\sinh n_2 a}.$$

§ 7. In the general formulæ of § 3 it is not necessary that the numerators should be expressed as sines and cosines, and the denominators as sinh's and cosh's; although, if a be real, the denominators must be of the form $\sinh ma$ or $\cosh ma$. In every equation of § 3 we may replace the sin's and cos's by sinh's and cosh's, and as the numerators and denominators are then expressed by a uniform notation, this is in some respects the better mode of writing the equations. Thus, taking for example the first equation, we have

$$\sum_{-\infty}^{\infty} \frac{\sinh n_1 x}{\sinh n_1 a} \times \sum_{-\infty}^{\infty} \frac{\cosh n_1 x}{\cosh n_1 a} = \sum_{-\infty}^{\infty} n_2 \frac{\sinh n_2 x}{\cosh n_2 a},$$

or

$$\sum_{-\infty}^{\infty} \frac{e^{n_1 a} - e^{-n_1 a}}{e^{n_1 x} - e^{-n_1 x}} \times \sum_{-\infty}^{\infty} \frac{e^{n_1 x} + e^{-n_1 x}}{e^{n_1 a} + e^{-n_1 a}} = \sum_{-\infty}^{\infty} n_2 \frac{e^{n_2 x} - e^{-n_2 x}}{e^{n_2 a} + e^{-n_2 a}},$$

subject to the condition that the real part of x must be numerically less than the real part of a, which is supposed positive.

We may also express the formulæ in a purely algebraical form; thus, taking the same example, we have

$$\sum_{-\infty}^{\infty} \frac{x^{2n-1}(1 - z^{4n-2})}{z^{2n-1}(1 - x^{4n-2})} \times \sum_{-\infty}^{\infty} \frac{x^{2n-1}(1 + z^{4n-2})}{z^{2n-1}(1 + x^{4n-2})}$$
$$= \sum_{-\infty}^{\infty} 2n \frac{x^{2n}(1 - z^{4n})}{z^{2n}(1 + x^{4n})},$$

where we must have $\operatorname{mod}\log z < \operatorname{mod}\log x$, so that, if $[x]$ and $[z]$ denote the analytical moduli of x and z, then the

greater of the two quantities $[z]$ and $\frac{1}{[z]}$ must be less than $\frac{1}{[x]}$, it being supposed that $[x]$ is less than unity.

§ 8. On looking at the formulæ in § 2 it will be noticed that the values of

$$\sum_{-\infty}^{\infty} \frac{\sin 2nx}{\sinh 2na} \quad \text{and} \quad \sum_{-\infty}^{\infty} (2n-1) \frac{\sin 2nx}{\cosh 2na}$$

seem required to complete the system. These series are connected with $Z(u)$ and $\operatorname{sn}^2 u$; and, putting $q = e^{-2a}$ as in § 2 and giving to ρ and u the same meanings as in § 1, we have the system of formulæ

$$\sum_{-\infty}^{\infty} \frac{\sin(2n-1)x}{\sinh(2n-1)a} = k\rho \operatorname{sn} u,$$

$$\sum_{-\infty}^{\infty} \frac{\cos(2n-1)x}{\cosh(2n-1)a} = k\rho \operatorname{cn} u,$$

$$\sum_{-\infty}^{\infty} \frac{\cos 2nx}{\cosh 2na} = \rho \operatorname{dn} u,$$

$$\sum_{-\infty}^{\infty} \frac{\sin 2nx}{\sinh 2na} = \rho Z(u),$$

$$\sum_{-\infty}^{\infty} (2n-1) \frac{\cos(2n-1)x}{\sinh(2n-1)a} = k\rho^2 \operatorname{cn} u \operatorname{dn} u,$$

$$\sum_{-\infty}^{\infty} (2n-1) \frac{\sin(2n-1)x}{\cosh(2n-1)a} = k\rho^2 \operatorname{sn} u \operatorname{dn} u,$$

$$\sum_{-\infty}^{\infty} (2n) \frac{\sin 2nx}{\cosh 2na} = k^2\rho^2 \operatorname{sn} u \operatorname{cn} u,$$

$$\sum_{-\infty}^{\infty} (2n) \frac{\cos 2nx}{\sinh 2na} = k^2\rho^2 \left\{ \frac{K-E}{k^2 K} - \operatorname{sn}^2 u \right\}.$$

It is obvious that as far as the groups of algebraical formulæ are concerned we are restricted to the first three formulæ of each group. The addition equation for the second Elliptic Integral, viz.

$$Z(u) + Z(v) + Z(w) = k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn} w,$$

where $u + v + w = 0$, gives rise however to an identical equation in which the first and fourth expressions are involved.

This relation may be written*

$$\sum_{-\infty}^{\infty} \frac{\sin n_1 x}{\sinh n_1 a} \times \sum_{-\infty}^{\infty} \frac{\sin n_1 y}{\sinh n_1 a} \times \sum_{-\infty}^{\infty} \frac{\sin n_1 z}{\sinh n_1 a}$$

$$= - \sum_{-\infty}^{\infty} \frac{n_1}{\sinh n_1 a} \times \sum_{-\infty}^{\infty} \frac{\sin n_2 x + \sin n_2 y + \sin n_2 z}{\sinh n_2 a},$$

where $x + y + z = 0$; or, replacing the sin's by sinh's, we have

$$\sum_{-\infty}^{\infty} \frac{\sinh (2n-1)\, x}{\sinh (2n-1)\, a} \times \sum_{-\infty}^{\infty} \frac{\sinh (2n-1)\, y}{\sinh (2n-1)\, a}$$

$$\times \sum_{-\infty}^{\infty} \frac{\sinh (2n-1)\, z}{\sinh (2n-1)\, a}$$

$$= \sum_{-\infty}^{\infty} \frac{2n-1}{\sinh (2n-1)\, a} \times \sum_{-\infty}^{\infty} \frac{\sinh 2nx + \sinh 2ny + \sinh 2nz}{\sinh 2na}$$

$$= 4 \sum_{-\infty}^{\infty} \frac{2n-1}{\sinh (2n-1)\, a} \times \sum_{-\infty}^{\infty} \frac{\sinh nx \sinh ny \sinh nz}{\sinh 2na},$$

where $x + y + z = 0$ and x, y, z must be such that the real part of each of them is numerically less than the real part of a.

ON THE DEVELOPMENT OF DETERMINANTS WHICH HAVE POLYNOMIAL ELEMENTS.

By *Thomas Muir, M.A., F.R.S.E.*

1. THE only development of determinants with polynomial elements which has been published and put to use is Albeggiani's. A lengthy paper is devoted to it by its author in the *Giornale di Matematica*, XIII. pp. 1–32, and Mr. Scott has given a condensed investigation of it with some deductions in his text-book (pp.38–40).

2. Let the determinant be of the n^{th} order, and let each element consist of p terms; then if we denote by D_1 the determinant whose elements are in order all the first terms of the polynomial, by D_2 the determinant whose elements are in order all the second terms, and so on, the development

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* *Messenger*, vol. xii. p. 48.

may be characterized as given in terms of $D_1, D_2, \ldots, D_p$ ai their minors. Thus, taking the determinant

$$\begin{vmatrix} a_{11}+b_{11}+c_{11}+d_{11} & a_{12}+b_{12}+c_{12}+d_{12} & a_{13}+b_{13}+c_{13}+d_{13} \\ a_{21}+b_{21}+c_{21}+d_{21} & a_{22}+b_{22}+c_{22}+d_{22} & a_{23}+b_{23}+c_{23}+d_{23} \\ a_{31}+b_{31}+c_{31}+d_{31} & a_{32}+b_{32}+c_{32}+d_{32} & a_{33}+b_{33}+c_{33}+d_{33} \end{vmatrix}$$

or $\Delta_{3,4}$ say, and expressing it by means of determinants wi single-termed elements, we have

$$\Delta_{3,4} = \delta_1 + \delta_2 + \delta_3 + \ldots + \delta_{64}$$

where

$$\delta_1 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \quad \delta_{22} = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix},$$

$$\delta_{43} = \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}, \quad \delta_{64} = \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix};$$

and the sixty others are of mixed constitution, that is to sa their elements are not all a's, or all b's, or all c's, or all d Now the object in view is to have in the development determinants of mixed constitution. Taking therefore t sixty in succession we use Laplace's expansion-theorem express each of them in terms of 'unmixed' determina Thus, instead of

$$\delta_{23}, \textit{ i.e. } \begin{vmatrix} b_{11} & b_{12} & c_{13} \\ b_{21} & b_{22} & c_{23} \\ b_{31} & b_{32} & c_{33} \end{vmatrix}$$

we write

$$c_{13}\begin{vmatrix} b_{21} & b_{22} \\ b_{31} & b_{32} \end{vmatrix} - c_{23}\begin{vmatrix} b_{11} & b_{12} \\ b_{31} & b_{32} \end{vmatrix} + c_{33}\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix};$$

instead of $$\delta_{25}, \textit{ i.e. } \begin{vmatrix} b_{11} & c_{12} & a_{13} \\ b_{21} & c_{22} & a_{23} \\ b_{31} & c_{32} & a_{33} \end{vmatrix}$$

we write

$$b_{11}c_{22}a_{33} + b_{21}c_{32}a_{13} + b_{31}c_{12}a_{23} - b_{31}c_{22}a_{13} - b_{21}c_{12}a_{33} - b_1$$

and so on. When this series of transformation completely effected, the result is Albeggiani's d of the given determinant $\Delta_{3,4}$.

3. It is hard to see that Albeggiani's *formula* for this development can be of much value; on one point however there can be little doubt, viz. that if it be the development itself which we want in any particular case, we can obtain it in the way above indicated with no less facility and with much more certainty as to accuracy of sign and non-omission of terms than by having recourse to the formula.

4. Another theorem regarding determinants with polynomial elements may be stated as follows:

Let $D_{n,p}$ be a determinant of the n^{th} order each of whose elements consists of p terms; let $\Sigma D_{n,p-1}$ denote the sum of the p determinants formed from $D_{n,p}$ by omitting firstly all the first terms of the elements, secondly all the second terms, and so on; let $\Sigma D_{n,p-2}$ denote the sum of the $\frac{1}{2}p(p-1)$ determinants formed by omitting firstly all the first and all the second terms of the elements of $D_{n,p}$, secondly all the first and all the third terms, and so on; and let $\Sigma D_{n,p-3}$, $\Sigma D_{n,p-4}$, &c. bear similar interpretations; then

$$D_{n,p} - \Sigma D_{n,p-1} + \Sigma D_{n,p-2} - \Sigma D_{n,p-3} + \ldots = 0. \quad (p > n) \ldots (\mathrm{I}).$$

To prove the truth of this we have only to think of $D_{p,p}$ as being partitioned into p^n determinants with monomial elements and to attend to one of these, say the first, δ_1, whose elements are in order all the first terms of the n^2 polynomials. If we then imagine all the other determinants, which are referred to in the theorem, partitioned in like manner, it is clear that the number of times δ_1 occurs

in $D_{n,p}$ is 1 time,

in $\Sigma D_{n,p-1}$ is $\frac{1}{2}(p-1)(p-2)$ times,

in $\Sigma D_{n,p-2}$ is $\frac{1}{6}(p-1)(p-2)(p-3)$ times,

....

Hence the coefficient of δ_1 in the left-hand member of the identity is

$$1 - C_{p-1,1} + C_{p-1,2} - C_{p-1,3} + \ldots;$$

i.e.

$$(1-1)^{p-1} \text{ or } 0,$$

as it should be. The same holds good of any other of the of the p^n determinants with monomial elements, and thus the identity is established.

5. Making all the terms vanish in all those elements of $D_{n,p}$ which lie outside the principal diagonal, we have the following theorem regarding row-multiplication.

If $\pi_{n,p}$ denote the product of n p-termed expressions; if $\Sigma\pi_{n,p-1}$ denote the sum of the p products formed from $\pi_{n,p}$ by omitting firstly all the first terms of the expressions, secondly all the second terms, and so on; if $\Sigma\pi_{n,p-2}$, $\Sigma\pi_{n,p-3}$, ... bear similar interpretations; then

$$\pi_{n,p} - \Sigma\pi_{n,p-1} + \Sigma\pi_{n,p-2} - \ldots = 0. \quad (p > n)\ldots\ldots(\text{II}).$$

6. Further by making all the polynomials in $\pi_{n,p}$ identical, there results the equality

$$\begin{aligned}(a_1+n_2+\ldots+a_p)^n - {}& \Sigma\,(a_1+a_2+\ldots+a_{p-1})^n \\ & + \Sigma\,(a_1+a_2+\ldots+a_{p-2})^n \\ & - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + (-1)^{p-1}\,\Sigma a_1^n \qquad = 0. \quad (p>n)\ldots(\text{III}).\end{aligned}$$

When $p = n$, the right-hand member is

$$(n\,!)\,a_1a_2\ldots a_n\,;$$

when $p = n - 1$, it is

$$\tfrac{1}{2}\,(n\,!)\ \Sigma a_1^2a_2\ldots a_{n-1}.$$

7. To illustrate these identities we may start with $\Delta_{2,4}$, and then specialize. Thus

$$\begin{aligned}
&\begin{vmatrix} a_1+b_1+c_1+d_1 & x_1+y_1+z_1+w_1 \\ a_2+b_2+c_2+d_2 & x_2+y_2+z_2+w_2 \end{vmatrix} \\
&\left.\begin{aligned}
-\begin{vmatrix} a_1+b_1+c_1 & x_1+y_1+z_1 \\ a_2+b_2+c_2 & x_2+y_2+z_2 \end{vmatrix} - \begin{vmatrix} a_1+b_1+d_1 & x_1+y_1+w_1 \\ a_2+b_2+d_2 & x_2+y_2+w_2 \end{vmatrix} \\
-\begin{vmatrix} a_1+c_1+d_1 & x_1+z_1+w_1 \\ a_2+c_2+d_2 & x_2+z_2+w_2 \end{vmatrix} - \begin{vmatrix} b_1+c_1+d_1 & y_1+z_1+w_1 \\ b_2+c_2+d_2 & y_2+z_2+w_2 \end{vmatrix}
\end{aligned}\right\} \\
&\left.\begin{aligned}
+\begin{vmatrix} a_1+b_1 & x_1+y_1 \\ a_2+b_2 & x_2+y_2 \end{vmatrix} + \begin{vmatrix} a_1+c_1 & x_1+z_1 \\ a_2+c_2 & x_2+z_2 \end{vmatrix} + \begin{vmatrix} a_1+d_1 & x_1+w_1 \\ a_2+d_2 & x_2+w_2 \end{vmatrix} \\
+\begin{vmatrix} b_1+c_1 & y_1+z_1 \\ b_2+c_2 & y_2+z_2 \end{vmatrix} + \begin{vmatrix} b_1+d_1 & y_1+w_1 \\ b_2+d_2 & y_2+w_2 \end{vmatrix} + \begin{vmatrix} c_1+d_1 & z_1+w_1 \\ c_2+d_2 & z_2+w_2 \end{vmatrix}
\end{aligned}\right\} \\
&-\begin{vmatrix} a_1 & x_1 \\ a_2 & x_2 \end{vmatrix} - \begin{vmatrix} b_1 & y_1 \\ b_2 & y_2 \end{vmatrix} - \begin{vmatrix} c_1 & z_1 \\ c_2 & z_2 \end{vmatrix} - \begin{vmatrix} d_1 & w_1 \\ d_2 & w_2 \end{vmatrix} = 0\,;
\end{aligned}$$

and putting $a_2 = b_2 = c_2 = d_2 = 0 = x_1 = y_1 = z_1 = w_1$, we have

$$
\begin{aligned}
&(a+b+c+d)(x+y+z+w) \\
&-(a+b+c,\ a+b+d,\ a+c+d,\ b+c+d) \\
&\qquad\qquad (x+y+z,\ x+y+w,\ x+z+w,\ y+z+w) \\
&+(a+b,\ a+c,\ a+d,\ b+c,\ b+d,\ c+d) \\
&\qquad\qquad (x+y,\ x+z,\ x+w,\ y+z,\ y+w,\ z+w) \\
&-(a,\ b,\ c,\ d \between x,\ y,\ z,\ w) \\
&= 0;
\end{aligned}
$$

and finally by making $x, y, z, w = a, b, c, d$ there results the identity expressing the sum of *seven* squares as the sum of *eight* squares, viz.

$$
\begin{aligned}
&(a+b+c+d)^2 \\
&+(a+b)^2+(a+c)^2+(a+d)^2+(b+c)^2+(b+d)^2+(c+d)^2 \\
=\ &(a+b+c)^2+(a+b+d)^2+(a+c+d)^2+(b+c+d)^2 \\
&+\ a^2\ +\ b^2\ +\ c^2\ +\ d^2.
\end{aligned}
$$

Beechcroft, Bishopton, N.B.
Nov. 24, 1883.

PROOF OF A THEOREM BY CAYLEY IN REGARD TO MATRICES.

By *A. R. Forsyth.*

IN a memoir on the Theory of Matrices (*Phil. Trans.* 1858, pp. 17–37) Prof. Cayley enunciates (but without proof for the general case) the theorem "that any matrix whatever satisfies an algebraical equation of its own order, the coefficient of the highest power being unity and those of the other powers functions of the terms of the matrix, the last coefficient being in fact the determinant." A proof of this is easily obtained as follows.

Consider, merely for conciseness of writing, a matrix of order three, say

$$M = \left(\begin{array}{ccc} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{array}\right).$$

1. To obtain the value of M^n.

Let it be denoted by

$$M^n = \begin{pmatrix} \alpha_n & \beta_n & \gamma_n \\ \alpha'_n & \beta'_n & \gamma'_n \\ \alpha''_n & \beta''_n & \gamma''_n \end{pmatrix};$$

then, defining $$M^{n+1} = M \cdot M^n,$$

we have

$$\begin{pmatrix} \alpha_{n+1} & \beta_{n+1} & \gamma_{n+1} \\ \alpha'_{n+1} & \beta'_{n+1} & \gamma'_{n+1} \\ \alpha''_{n+1} & \beta''_{n+1} & \gamma''_{n+1} \end{pmatrix} = \begin{pmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{pmatrix} \begin{pmatrix} \alpha_n & \beta_n & \gamma_n \\ \alpha'_n & \beta'_n & \gamma'_n \\ \alpha''_n & \beta''_n & \gamma''_n \end{pmatrix}.$$

Expanding the right-hand side, we obtain the equations

$$\alpha_{n+1} = a\,\alpha_n + b\,\alpha'_n + c\,\alpha''_n,$$
$$\alpha'_{n+1} = a'\,\alpha_n + b'\,\alpha'_n + c'\,\alpha''_n,$$
$$\alpha''_{n+1} = a''\alpha_n + b''\alpha'_n + c''\alpha''_n.$$

Let ξ, η, ζ be the roots of

$$\begin{vmatrix} a-x & b & c \\ a' & b'-x & c' \\ a'' & b'' & c''-x \end{vmatrix}$$
$$= P_3 - P_2 x + P_1 x^2 - x^3 = 0.$$

Then the foregoing difference equations at once show that α_n, α'_n, α''_n are all of the form

$$\xi^n.\text{const.} + \eta^n.\text{const.} + \zeta^n.\text{const.},$$

the values of the constants varying from one constituent of the matrix to another; they are not independent but are connected by relations arising from the difference equations. Thus, writing

$$\alpha_n^{(n)} = A^{(n)}\xi^n + B^{(n)}\eta^n + C^{(n)}\zeta^n,$$

then
$$A\,\xi = a\,A + b\,A' + c\,A'',$$
$$A'\,\xi = a'\,A + b'\,A' + c'\,A'',$$
$$A''\xi = a''A + b''A' + c''A'';$$

and therefore A, A', A'' are in the ratio of the minors of the above vanishing determinant when ξ is substituted for x. Similarly for the B's with η substituted for x, and for the C's with ζ.

To determine A, B, C (and so all the constants) the equations are

(i) $n=0$, $\qquad 1=A+B+C$.

(ii) $n=1$, $\qquad a=A\xi+B\eta+C\zeta$,

$$a^2+a'b+a''c=A\xi^2+B\eta^2+C\zeta^2;$$

and therefore

$$A(\xi-\eta)(\eta-\zeta)(\zeta-\xi)=(\zeta-\eta)[\eta\zeta-a(\eta+\zeta)+a^2+a'b+a''c].$$

Now the equation giving x is

$$x^3-x^2(a+b'+c'')+x(ab'+b'c''+c''a-a'b-b''c'-ca'')-\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}=0;$$

and by means of the relations between the roots and coefficients we easily obtain

$$\eta\zeta-a(\eta+\zeta)+a^2+a'b+a''c=\begin{vmatrix} b'-\xi & c \\ b'' & c''-\xi \end{vmatrix}.$$

Hence

$$\frac{A}{\begin{vmatrix} b'-\xi & c' \\ b'' & c''-\xi \end{vmatrix}}=\frac{A'}{\begin{vmatrix} c' & a' \\ c''-\xi & a'' \end{vmatrix}}=\frac{A''}{\begin{vmatrix} a' & b'-\xi \\ a'' & b'' \end{vmatrix}}$$

$$=\frac{1}{(\xi-\eta)(\xi-\zeta)}=\frac{1}{3\xi^2-2P_1\xi+P_2},$$

and similar equations for B with η substituted for ξ, and for C with ζ substituted for ξ.

2. If the matrix M be looked upon as a single quantity involving the matrix unity, then it satisfies the equation

$$\begin{vmatrix} a-M & b & c \\ a' & b'-M & c' \\ a'' & b'' & c''-M \end{vmatrix}=0,$$

For, expanding the left-hand side we have it at once equal to

$$P_3-P_2M+P_1M^2-M^3.$$

Let this be combined into a single matrix according to the elementary rules applying thereto, then it comes to be

of the form

$$\begin{pmatrix} P_3-P_2\alpha_1+P_1\alpha_2-\alpha_3 & P_3-P_2\beta_1+P_1\beta_2-\beta_3 & P_3-P_2\gamma_1+P_1\gamma_2-\gamma_3 \\ P_3-P_2\alpha'_1+P_1\alpha'_2-\alpha'_3 & P_3-P_2\beta'_1+P_1\beta'_2-\beta'_3 & P_3-P_2\gamma'_1+P_1\gamma'_2-\gamma'_3 \\ P_3-P_2\alpha''_1+P_1\alpha''_2-\alpha''_3 & P_3-P_2\beta''_1+P_1\beta''_2-\beta''_3 & P_3-P_2\gamma''_1+P_1\gamma''_2-\gamma''_3 \end{pmatrix},$$

and this has to be proved a zero-matrix.

Consider the first element: it is

$$P_3-P_2(A\xi+B\eta+C\zeta)+P_1(A\xi^2+B\eta^2+C\zeta^2)-(A\xi^3+B\eta^3+C\zeta^3)$$
$$=P_3-P_3(A+B+C)=0,$$

since $A+B+C=1$; similarly all the other elements are zero and the above is therefore a zero-matrix. Hence M satisfies the equation of its own order.

It is evident that the above work could be reproduced line by line with n^2 constituents instead of 3^2, and that it would then furnish for the most general matrix a proof of Cayley's theorem.

NOTE ON AN ALGEBRAIC IDENTITY.

By *Capt. P. A. MacMahon, R.A.*

THE present note relates to Mr. Prior's algebraic identity that has been considered by Mr. J. W. L. Glaisher in the *Messenger* for 1880. It is, that when $a+b+c=0$,

$$\left(\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}\right)\left(\frac{b-c}{a}+\frac{c-a}{b}+\frac{a-b}{c}\right)=9.$$

I proceed to shew that a similar theorem exists in the case of any number of quantities, subject to certain conditions. Writing a_1, a_2, a_3 for a, b, c, the identity may be written

$$\left(\frac{a_1}{a_2-a_3}-\frac{a_2}{a_1-a_3}+\frac{a_3}{a_1-a_2}\right)\left(\frac{a_2-a_3}{a_1}-\frac{a_1-a_3}{a_2}+\frac{a_1-a_2}{a_3}\right)=3^2.$$

Wherein now the suffixes, in the differences involved, are in numerical order.

In the case of two quantities, if $a_1+a_2=0$, it is easy to verify that

$$(a_1-a_2)\left(\frac{1}{a_1}-\frac{1}{a_2}\right)=2^2.$$

For four quantities I find that

$$\left\{\frac{a_1}{(a_2-a_3)(a_2-a_4)(a_3-a_4)}-\frac{a_2}{(a_1-a_3)(a_1-a_4)(a_3-a_4)}\right.$$
$$\left.+\frac{a_3}{(a_1-a_2)(a_1-a_4)(a_2-a_4)}-\frac{a_4}{(a_1-a_2)(a_1-a_3)(a_2-a_3)}\right\}$$
$$\times\left\{\frac{(a_2-a_3)(a_2-a_4)(a_3-a_4)}{a_1}-\frac{(a_1-a_3)(a_1-a_4)(a_3-a_4)}{a_2}\right.$$
$$\left.+\frac{(a_1-a_2)(a_1-a_4)(a_2-a_4)}{a_3}-\frac{(a_1-a_2)(a_1-a_3)(a_2-a_3)}{a_4}\right\}=4^2,$$

subject to the two conditions

$$a_1+a_2+a_3+a_4=0,$$
$$a_1^2+a_2^2+a_3^2+a_4^2=0.$$

This may be proved as follows:—

The left-hand side may be written

$$\frac{\Sigma a_1(a_1-a_2)(a_1-a_3)(a_1-a_4)}{(a_1-a_2)(a_1-a_3)(a_1-a_4)(a_2-a_3)(a_2-a_4)(a_3-a\ )}$$
$$\times\frac{a_2a_3a_4(a_2-a_3)(a_2-a_4)(a_3-a_4)-a_1a_3a_4(a_1-a_3))a_1-a_4)(a_3-a_4)+\ldots}{a_1a_2a_3a_4}.$$

Now $\Sigma(a_1-a_2)(a_1-a_3)(a_1-a_4)$

$$=\Sigma\{a_1^4-(a_2+a_3+a_4)a_1^3+(a_3a_4+a_4a_2+a_2a_3)a_1^2-a_1a_2a_3a_4\}$$
$$=\Sigma(a_1^4+a_1^4+a_1^4-a_1a_2a_3a_4)$$
$$=3(a_1^4+a_2^4+a_3^4+a_4^4)-4a_1a_2a_3a_4,$$

but if a_1, a_2, a_3, a_4 be roots of the quartic

$$x^4-dx+e=0,$$
$$a_1^4+a_2^4+a_3^4+a_4^4=-4e=-12a_1a_2a_3a_4;$$

consequently

$$\Sigma a_1(a_1-a_2)(a_1-a_3)(a_1-a_4)=-16a_1a_2a_3a_4;$$

also since the expression

$$a_2a_3a_4(a_2-a_3)(a_2-a_4)(a_3-a_4)-a_1a_3a_4(a_1-a_3)(a_1-a_4)(a_3-a_4),$$
$$+a_1a_2a_4(a_1-a_2)(a_1-a_4)(a_2-a_4)-a_1a_2a_3(a_1-a_2)(a_1-a_3)(a_2-a_3)$$

vanishes for each of the substitutions

$$a_1=a_2,\ a_1=a_3,\ a_1=a_4,\ a_2=a_3,\ a_2=a_4,\ a_3=a_4,$$

we may put it equal to

$$m(a_1-a_2)(a_1-a_3)(a_1-a_4)(a_2-a_3)(a_2-a_4)(a_3-a_4),$$

and m is evidently -1; thus the left-hand side is $+16$ as we wished to prove.

Suppose now we have n quantities

$$a_1,\ a_2,\ a_3,\ \ldots a_n,$$

then if

$$\left.\begin{array}{l} a_1\ +a_2\ +a_3\ +\ldots+a_n\ =0 \\ a_1^2\ +a_2^2\ +a_3^2\ +\ldots+a_n^2\ =0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_1^{n-2}+a_2^{n-2}+a_3^{n-2}+\ldots+a_n^{n-2}=0 \end{array}\right\},$$

clearly

$$\begin{aligned} &\Sigma a_1(a_1-a_2)(a_1-a_3)\ldots(a_1-a_n) \\ &=(n-1)(a_1^n+a_2^n+a_3^n+\ldots+a_n^n)+(-)^{n+1}na_1a_2a_3\ldots a_n \\ &=(-)^{n+1}\{n(n-1)+n\}a_1a_2a_3\ldots a_n \\ &=(-)^{n+1}n^2a_1a_2a_3\ldots a_n; \end{aligned}$$

also, as before

$$\begin{aligned} &a_2a_3a_4\ldots a_n(a_2-a_3)(a_2-a_4)\ldots(a_2-a_n)(a_3-a_n)\ldots(a_3-a_n) \\ &\qquad\ldots(a_{n-1}-a_n)-a_1a_3a_4\ldots a_n(a_1-a_3)(a_1-a_4)\ldots(a_1-a_n), \\ &(a_3-a_4)(a_3-a_5)\ldots(a_3-a_n)\ldots(a_{n-1}-a_n)+\ldots \\ &=(-)^{n+1}(a_1-a_2)(a_1-a_3)\ldots(a_1-a_n)(a_2-a_3)\ldots(a_2-a_n)\ldots(a_{n-1}-a_n). \end{aligned}$$

Consequently we have proved the generalised identity:—

$$\begin{aligned} &\left[\frac{a_1}{(a_2-a_3)\ldots(a_2-a_n)(a_3-a_4)\ldots(a_3-a_n)\ldots(a_{n-1}-a_n)}\right. \\ &\left.-\frac{a_2}{(a_1-a_3)\ldots(a_1-a_n)(a_3-a_4)\ldots(a_3-a_n)\ldots(a_{n-1}-a_n)}+\ldots\right] \\ &\times\left[\frac{(a_2-a_3)\ldots(a_2-a_n)(a_3-a_4)\ldots(a_3-a_n)\ldots(a_{n-1}-a_n)}{a_1}\right. \\ &\left.-\frac{(a_1-a_3)\ldots(a_1-a_n)(a_3-a_4)\ldots(a_3-a_n)\ldots(a_{n-1}-a_n)}{a_2}+\ldots\right]=n^2. \end{aligned}$$

Royal Military Academy,
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ON A CUBIC SURFACE.

By *H. M. Taylor, M.A.*

It is a well-known theorem in plane geometry, that the locus of the centres of all conics, which pass through four fixed points, *i.e.* the intersections of two conics, is a conic, which bisects the distances between each pair of points.

To this corresponds the following theorem in space, the locus of the centres of all quadrics, which pass through the eight intersections of three given quadrics, is a cubic surface, which bisects the distances between each pair of points of intersection.

Let the equations of the three quadrics be

$$S \equiv ax^2 + by^2 + cz^2 + 2lyz + 2mzx + 2nxy + 2fx + 2gy + 2hz - 1 = 0,$$

$$S' = 0,$$

$$S'' = 0;$$

and let $\frac{dS}{dx} = P, \quad \frac{dS}{dy} = Q, \quad \frac{dS}{dz} = R, \quad$ &c.

The equation of any quadric through the intersections of $S = 0$, $S' = 0$, $S'' = 0$ may be written

$$S + \lambda S' + \mu S'' = 0,$$

and its centre will be found from the equations

$$P + \lambda Q + \mu R = 0,$$

$$P' + \lambda Q' + \mu R' = 0,$$

$$P'' + \lambda Q'' + \mu R'' = 0.$$

The locus of the centre will be the surface whose equation is

$$\begin{vmatrix} P, & Q, & R \\ P', & Q', & R' \\ P'', & Q'', & R'' \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots (1),$$

which, written at length, is

$$\begin{vmatrix} ax + ny + mz + f, & a'x+n'y+m'z+f', & a''x+n''y+m''z+f'' \\ nx + by + lz + g, & n'x+b'y+l'z+g', & n''x+b''y+l''z+g'' \\ mx + ly + cz + h, & m'x+l'y+c'z+h', & m''x+l''y+c''z+h'' \end{vmatrix} = 0.$$

By the theory of determinants, multiplying the first, second, and third rows by $2(x_1-x_2)$, $2(y_1-y_2)$, $2(z_1-z_2)$ respectively, and adding for a new first row,

$$\begin{vmatrix} 2(x_1-x_2)P+2(y_1-y_2)Q+2(z_1-z_2)R\,, & Q\,, & R \\ 2(x_1-x_2)P'+2(y_1-y_2)Q'+2(z_1-z_2)R'\,, & Q'\,, & R' \\ 2(x_1-x_2)P''+2(y_1-y_2)Q''+2(z_1-z_2)R''\,, & Q''\,, & R'' \end{vmatrix}=0.$$

Now, substituting $\frac{1}{2}(x_1+x_2)$, $\frac{1}{2}(y_1+y_2)$, $\frac{1}{2}(z_1+z_2)$ for x, y, z respectively, we obtain

$$\begin{aligned} &2(x_1-x_2)P+2(y_1-y_2)Q+2(z_1-z_2)R \\ &= \{a(x_1+x_2)+n(y_1+y_2)+m(z_1+z_2)+2f\}(x_1-x_2) \\ &\quad+\{n(x_1+x_2)+b(y_1+y_2)+l\,(z_1+z_2)+2g\}(y_1-y_2) \\ &\quad+\{m(x_1+x_2)+l(y_1+y_2)+c\,(z_1+z_2)+2h\}(z_1-z_2), \\ &= a(x_1^2-x_2^2)+b(y_1^2-y_2^2)+c(z_1^2-z_2^2) \\ &\quad+l\,\{(y_1+y_2)(z_1-z_2)+(z_1+z_2)(y_1-y_2)\} \\ &\quad+m\,\{(z_1+z_2)(x_1-x_2)+(x_1+x_2)(z_1-z_2)\} \\ &\quad+n\,\{(x_1+x_2)(y_1-y_2)+(y_1+y_2)(x_1-x_2)\} \\ &\quad+2f(x_1-x_2)+2g(y_1-y_2)+2(z_1-z_2), \\ &= a(x_1^2-x_2^2)+b(y_1^2-y_2^2)+c(z_1^2-z_2^2) \\ &\quad+2l(y_1z_1-y_2z_2)+2m(z_1x_1-z_2x_2)+2n(x_1y_1-x_2y_2) \\ &\quad+2f(x_1-x_2)+2g(y_1-y_2)+2h(z_1-z_2), \\ &= S_1-S_2=0, \text{ if } (x_1, y_1, z_1), (x_2, y_2, z_2) \text{ are points on } S=0. \end{aligned}$$

Similarly,

$$2(x_1-x_2)P'+2(y_1-y_2)Q'+2(z_1-z_2)R'=S_1'-S_2'=0,$$

if (x_1, y_1, z_1), (x_2, y_2, z_2) are points on $S'=0$; and

$$2(x_1-x_2)P''+2(y_1-y_2)Q''+2(z_1-z_2)R''=S_1''-S_2''=0,$$

if (x_1, y_1, z_1), (x_2, y_2, z_2) are points on $S''=0$.

Therefore the coordinates

$$x=\tfrac{1}{2}(x_1+x_2),\quad y=\tfrac{1}{2}(y_1+y_2),\quad z=\tfrac{1}{2}(z_1+z_2),$$

satisfy the equation of the cubic surface (1) under the conditions that (x_1, y_1, z_1), (x_2, y_2, z_2) are points of intersection of the three surfaces

$$S=0, \quad S'=0, \quad S''=0,$$

which proves the theorem.

As is well known, all quadrics which pass through seven fixed points pass also through an eighth. The above theorem might therefore be stated with respect to seven arbitrary fixed points, instead of with respect to the eight intersections of three given quadrics.

A NOTE ON ASSOCIATED FUNCTIONS AND SPHERICAL HARMONICS.

By *A. B. Basset.*

1. HEINE in his *Handbuch der Kugelfunctionen*, obtains expressions for these functions in the forms of definite integrals, by means of the properties of hyper-geometric series; the object of this note is to deduce these integrals by an elementary method.

2. If $$(1-2\mu x+x^2)^{-\frac{1}{2}(2m+1)}=\Sigma\Omega_r x^r,$$

it is known that Ω_r satisfies the differential equation

$$\frac{d}{d\mu}\left\{(1-\mu^2)^{m+1}\frac{d\Omega_r}{d\mu}\right\}+(2m+r+1)(2m+r+2)(1-m^2)^m\,\Omega_r=0,^*$$

and that $$\int_{-1}^{1}(1-\mu^2)^m\,\Omega_r\Omega_s d\mu=0,$$

provided r and s are different integers. To evaluate this integral when $r=s$, we have

$$\int_{-1}^{1}\frac{(1-\mu^2)^m\,d\mu}{(1-2\mu x+x^2)^{2m+1}}=\int_{-1}^{1}(1-\mu^2)^m(1+\Omega_1 x+\ldots\Omega_r x^r+\ldots)^2\,d\mu$$

$$\int_{-1}^{1}(1-\mu^2)^m(1+\Omega_1^2x^2+\ldots\Omega_r^2x^{2r}+\ldots)\;d\mu,$$

the required integral is therefore equal to the coefficient of x^{2r} in the integral on the left-hand side.

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* See a note by Mr. R. R. Webb, in the *Messenger of Mathematics*, vol. IX. p. 125.

Now if $\alpha > 1$, and $z = \dfrac{\alpha+1}{\alpha-1}\tan^2\tfrac{1}{2}\theta$,

$$\int_0^\pi \frac{\sin^{s-1}\theta\,d\theta}{(\alpha-\cos\theta)^s} = \frac{2^{s-1}}{(\alpha^2-1)^{\frac{1}{2}s}}\int_0^\infty \frac{z^{\frac{1}{2}s-1}dz}{(1+z)^s}$$
$$= \frac{2^{s-1}}{(\alpha^2-1)^{\frac{1}{2}s}}\cdot\frac{(\Gamma\tfrac{1}{2}s)^2}{\Gamma s}.$$

Integrating with respect to α between the limits ∞ and $\dfrac{1+x^2}{2x}$, we obtain

$$\int_0^\pi \frac{\sin^{s-1}\theta\,d\theta}{\left(\dfrac{1+x^2}{2x}-\cos\theta\right)^{s-1}} = \frac{2^{s-1}(\Gamma\tfrac{1}{2}s)^2}{\Gamma(s-1)}\int_{\frac{1+x^2}{2x}}^\infty \frac{d\alpha}{(\alpha^2-1)^{\frac{1}{2}s}}.$$

Putting $y = \alpha - \sqrt{(\alpha^2-1)}$ the integral

$$= \frac{2^{2s-3}(\Gamma\tfrac{1}{2}s)^2}{\Gamma(s-1)}\int_0^x \frac{y^{s-1}dy}{(1-y^2)^{s-1}}.$$

Hence if $s = 2m+2$, $\mu = \cos\theta$,

$$\int_{-1}^1 \frac{(1-\mu^2)^m\,d\mu}{(1-2x\mu+x^2)^{2m+1}} = \frac{2^{2m+1}(m\,!)^2}{x^{2m+1}2m\,!}\int_0^x \frac{y^{2m}dy}{(1-y^2)^{2m+1}};$$

therefore

$$\int_{-1}^1 (1-\mu^2)^m\,\Omega_r^2\,d\mu = \frac{2^{2m+1}(m\,!)^2(2m+1)(2m+2)\ldots(2m+r)}{(2m+2r+1)\,2m\,!}.$$

3. The connection between the associated functions P_n^m of the first kind, and the functions Ω is easily seen; for

$$(1-2\mu x+x^2)^{-\frac{1}{2}} = \Sigma P_n x^n;$$

therefore

$$\frac{1.3\ldots(2m-1)}{(1-2\mu x+x^2)^{\frac{1}{2}(2m+1)}} = \Sigma\frac{d^mP_n}{d\mu^m}x^{n-m}$$
$$= 1.3\ldots(2m-1)\,\Sigma\Omega_{n-m}x^{n-m},$$

where $n > m$; therefore

$$P_n^m = (1-\mu^2)^{\frac{1}{2}m}\frac{d^mP_n}{d\mu^m} = 1.3\ldots(2m-1)\,(1-\mu^2)^{\frac{1}{2}m}\Omega_{n-m}.$$

Hence

$$\int_{-1}^{1} (P_n^m)^2\,d\mu = \frac{1.3^2\ldots(2m-1)^2\,2^{2m+1}\,(2m+1)\,(2m+2)\ldots(n+m)}{(n-m)\,!\,2m\,!\,(2n+1)}$$

$$= \frac{2}{2n+1}\,\frac{(n+m)\,!}{(n-m)\,!}.$$

4. The definite integral which we have employed in Art. 2 enables us at once to deduce the value of P_n^m in a similar form, for if in the identity

$$\int_0^\pi \frac{\sin^{2m}\theta\,d\theta}{(a-b\cos\theta)^{2m+1}} = \frac{2^{2m}\,\Gamma^2\tfrac{1}{2}(m+1)}{(a^2-b^2)^{\frac{1}{2}(2m+1)}\,\Gamma(2m+1)}$$

we put $a=1-\mu x$, $b=x\sqrt{(\mu^2-1)}$, it becomes

$$\int_0^\pi \frac{\sin^{2m}\theta\,d\theta}{[1-x\{\mu+\sqrt{(\mu^2-1)}\cos\theta\}]^{2m+1}}$$

$$= \frac{2^{2m}\,\Gamma^2\tfrac{1}{2}(m+1)}{2m\,!}\cdot\frac{1}{(1-2\mu x+x^2)^{\frac{1}{2}(2m+1)}};$$

therefore, equating coefficients of x^{n-m},

$$\frac{(2m+1)(2m+2)\ldots(m+n)}{(n-m)\,!}\int_0^\pi \{\mu+\sqrt{(\mu^2-1)}\cos\theta)^{n-m}\sin^{2m}\theta\,d\theta$$

$$= \frac{2^{2m}\,\Gamma^2\tfrac{1}{2}\,(m+1)}{2m\,!}\,\Omega_{n-m};$$

therefore

$$P_n^m = \frac{(n+m)\,!\,(1-\mu^2)^{\frac{1}{2}m}}{(n-m)\,!\,1.3\ldots(2m-1)\pi}\int_0^\pi \{\mu+\sqrt{(\mu^2-1)}\cos\theta\}^{n-m}\sin^{2m}\theta\,d\theta.$$

If we put

$$\cos\phi = \frac{\mu\cos\theta+\sqrt{(\mu^2-1)}}{\mu+\sqrt{(\mu^2-1)}\cos\theta},$$

the definite integral is easily seen to be equal to

$$\int_0^\pi \frac{\sin^{2m}\phi\,d\phi}{\{\mu+\sqrt{(\mu^2-1)}\cos\theta\}^{n+m+1}},$$

and by means of Jacobi's formula

$$\cos m\phi = \frac{(-1)^m}{1.3\ldots(2m-1)}\,\frac{d^m}{dz^m}\,(1-z^2)^{m-\frac{1}{2}}\sin\phi$$

we can express P_n^m in either of the forms

$$\frac{(n+m)!}{n!\,\pi}\int_0^\pi \{\mu+\sqrt{(\mu^2-1}\cos\theta)\}^n\cos m\theta\,d\theta$$

or

$$\frac{(-1)^m n!}{(n-m)!\,\pi}\int_0^\pi \frac{\cos m\theta\,d\theta}{\{\mu+\sqrt{(\mu^2-1)}\cos\theta\}^{n+1}}$$

(see Heine, p. 215).

5. We shall now obtain the expression for $Q_n(v)$ given by Heine on page 132.

Let $$U=\frac{1}{2H}\log\frac{v+x+H}{v+x-H},$$

where $$H^2=1+2vx+x^2 \text{ and } v>1,$$

Then

$$(1-v^2)\frac{dU}{dv}=\frac{1}{H^2}\{1+vx-x(1-v^2)U\},$$

$$\frac{d}{dv}(1-v^2)\frac{dU}{dv}=\frac{x}{H}\{x^2-vx-2+(v^2x+2vx^2+2v+3x)U\}.$$

Also

$$\frac{d}{dx}(xU)=\frac{1}{H^2}\{x+(1+vx)U\},$$

$$x\frac{d^2}{d^2x}(xU)=\frac{x}{H^4}\{2+vx-x^2-(v^2x+2vx^2+2v+3x)U\};$$

therefore $$\frac{d}{dv}(1-v^2)\frac{dU}{dv}+v\frac{d^2}{dx^2}(xU)=0.$$

Hence if $U=\Sigma S_n x^n$, S_n satisfies the equation

$$\frac{d}{dv}(1-v^2)\frac{dS_n}{dv}+n(n+1)S_n=0;$$

and since $v>1$, S_n must be equal to $A_n Q_n(v)$ when A_n is some constant. Now if $a>b$,

$$\int_0^\infty \frac{d\theta}{a+b\cosh\theta}=\frac{1}{\sqrt{(a^2-b^2)}}\log\frac{a+\sqrt{(a^2-b^2)}}{b}.$$

Putting $a = v + x$, $b = \sqrt{(v^2 - 1)}$,

$$\int_0^\infty \frac{d\theta}{v + \sqrt{(v^2-1)}\cosh\theta + x} = \frac{1}{(1+2vx+x^2)^{\frac{1}{2}}} \log \frac{v+x+\sqrt{(1+2vx+x^2)}}{\sqrt{(v^2-1)}}$$

$$= \frac{1}{2H} \log \frac{v+x+H}{v+x-H}$$

$$= \Sigma A_n Q_n x^n.$$

Expanding the definite integral and equating the coefficients of x^n, we find that

$$(-1)^n A_n Q_n = \int_0^\infty \frac{d\theta}{\{v + \sqrt{(v^2-1)}\cosh\theta\}^{n+1}}.$$

If the left-hand side be expanded in powers of $\frac{1}{v}$, the coefficient of $\frac{1}{v^{n+1}}$—the first term is the expansion—is evidently equal to

$$\int_0^\infty \frac{d\theta}{(1+\cosh\theta)^{n+1}} = \int_0^1 (1-z)^n z^{-\frac{1}{2}} dz$$

$$= \frac{n!}{1.3\ldots(2n+1)};$$

comparing this with the series for Q_n (Ferrers' Spherical Harmonics, p. 105), we see that $A_n = (-1)^n$, whence

$$Q_n = \int_0^\infty \frac{d\theta}{\{v + \sqrt{(v^2-1)}\cosh\theta\}^{n+1}}.$$

6. To find the value of the associated functions of the second kind, let

$$V_m = \int_0^\infty \frac{\sinh^{2m}\theta\, d\theta}{\{v + \sqrt{(v^2-1)}\cosh\theta\}^{n+m+1}}, \qquad n > m;$$

therefore

$$\frac{dV_m}{dv} = -\frac{n+m+1}{\sqrt{(v^2-1)}} \int_0^\infty \frac{\{\sqrt{(v^2-1)} + v\cosh\theta\} \sinh^{2m}\theta\, d\theta}{\{v + \sqrt{(v^2-1)}\cosh\theta\}^{n+m+2}}$$

$$= -\frac{n+m+1}{\sqrt{(v^2-1)}} \int_0^\infty \frac{\sinh^{2m}\theta \cosh\theta\, d\theta}{\{v + \sqrt{(v^2-1)}\cosh\theta\}^{n+m+1}} + (n+m+1) V_{m+1}.$$

Integrating by parts, we find

$$\frac{dV_m}{dv} = -\frac{(n+m+1)(n-m)}{2m-1} V_{m+1}.$$

Now $V_0 = Q_n$, hence

$$\frac{d^m Q_n}{dv} = \frac{(-1)^m (n-m+1)(n-m+2)\dots(n+m)}{1.3\dots 2m-1} V_m;$$

therefore if Q_n^m denote the associated function of the second kind

$$Q_n^m = (v^2-1)^{\frac{1}{2}m} \frac{d^m Q_n}{dv^m}$$

$$= \frac{(-1)^m (n+m)!\,(v^2-1)^{\frac{1}{2}m}}{(n-m)!\,1.3\dots(2m-1)} \int_0^\infty \frac{\sinh^{2m}\theta\, d\theta}{\{v+\sqrt{(v^2-1)}\cosh\theta\}^{n+m+1}},$$

which by Jacobi's formula

$$= \frac{n!}{(n-m)!} \int_0^\infty \frac{\cosh m\theta\, d\theta}{\{v+\sqrt{(v^2-1)}\cosh\theta\}^{n+1}}.$$

11, Old Square,
Lincoln's Inn,
January 10*th*, 1884.

NOTE ON AN ELECTROSTATICAL PROBLEM.

By *Lionel Wilberforce, B.A.*, Scholar of Trinity College, Cambridge.

A CONSIDERABLE number of electrostatical problems relating to conducting spheres intersecting at angles which are sub-multiples of two right angles can be more easily solved directly by the method of images than by inversion of a case in which some of these spheres are replaced by planes.

We may solve in a similar manner certain cases where such a conductor is kept at zero potential under the action of an electrified point, and the method will occasionally be preferable to the more usual one of inverting a system consisting of a conductor of the same nature charged and in free space.

The problem at the end of § 168, Maxwell's *Electricity and Magnetism*, gives an example of the method, interesting as involving a geometrical theorem whose converse is akin to Euclid, VI. *D.*

The theorem is as follows:

Let A, B, O (fig. 5) be any three points, then, if in AO, BO two points F, E be taken so that

$$AF.AO + BE.BO = AB^2,$$

a circle can be described through O, F, E, and C the point of intersection of AE and BF.

To prove this, describe a circle through O, F, E, and let its centre be S and radius r. Join SB and draw AN perpendicular to it.

Then $\quad AF.AO + BE.BO = AS^2 + BS^2 - 2r^2,$

and $\quad AB^2 = AS^2 + SB^2 - 2.BS.SN.$

Hence $\quad BS.SN = r^2,$

that is, A lies on the polar of B with regard to the circle.

Hence evidently C lies on the circle.

The converse theorem follows at once from the same considerations.

Now consider two spheres, centres A, B, radii α, β, (fig. 6) cutting orthogonally and kept at zero potential under the action of an external electrified point O, whose charge is $-e$.

Let F, E be the images of O in the two spheres, and let AE, BF meet in C.

since $\quad AF.AO = \alpha^2, \quad BE.BO = \beta^2, \quad AB^2 = \alpha^2 + \beta^2,$

a circle can be described through O, F, C, E.

Hence $\quad AC.AE = AF.AO = \alpha^2,$

and $\quad BC.BF = BE.BO = \beta^2,$

and C is the image of E in the first sphere and of F in the second.

Let P be any point on the first sphere, join PB, PE, PO, and let $OA = a$, $OB = b$, $OP = r$, $BP = p$.

The image at F is $e\dfrac{\alpha}{a}$,

that at E is $e\dfrac{\beta}{b}$,

that at C is $-e\dfrac{\alpha}{a}\dfrac{\beta}{BF} = -e\dfrac{\beta}{b}\dfrac{\alpha}{AE}$, clearly.

Hence the density at P is

$$\frac{e}{4\pi\alpha}\frac{a^2-\alpha^2}{r^3}-\frac{e\beta}{b}\frac{1}{4\pi\alpha}\frac{AE^2-\alpha^2}{PE^3}.$$

To find PE, we have,

$$\frac{p^2+\frac{\beta^4}{b^2}-PE^2}{\frac{\beta^2}{b}}=\frac{p^2+b^2-r^2}{b},$$

or $$p^2\left(b-\frac{\beta^2}{b}\right)+\frac{\beta^4}{b}-\beta^2 b+r^2\frac{\beta^2}{b}=b.PE^2.$$

Therefore $(p^2-\beta^2)(b^2-\beta^2)+r^2\beta^2=b^2.PE^2.$

Write for p^2, $\alpha^2+\beta^2$, and for r^2, a^2, and we get

$$a^2\beta^2+b^2\alpha^2-\alpha^2\beta^2=b^2.AE^2.$$

Hence the images are $e\frac{\alpha}{a}$, $e\frac{\beta}{b}$, $-e\frac{\alpha\beta}{\sqrt{(a^2\beta^2+b^2\alpha^2-\alpha^2\beta^2)}}$.
The whole charge of the conductor is the sum of these.
The density at P is

$$\frac{e}{4\pi\alpha}\frac{a^2-\alpha^2}{r^3}\left\{1-\frac{\beta^3 r}{[r^2\beta^2+(p^2-\beta^2)(b^2-\beta^2)]^{\frac{3}{2}}}\right\}.$$

THEOREM RELATING TO THE SUM OF SELECTED BINOMIAL-THEOREM COEFFICIENTS.

By Prof. *P. G. Tait.*

LET equal masses be placed, two and two together, at the corners of an m-sided polygon. Slide one from each end of a side till they meet at its middle point. They now form a new, and smaller, m-sided polygon, but their centre of inertia has not been disturbed. Repeat the process indefinitely, and the masses will ultimately be collected in the centre of inertia.

Now if the distances of the corners of the original polygon from a fixed plane be

$$u_1, \quad u_2, \quad \ldots, \quad u_m,$$

those of the first derived polygon will be

$$\tfrac{1}{2}(u_1+u_2), \quad \tfrac{1}{2}(u_2+u_3), \quad \ldots, \quad \tfrac{1}{2}(u_m+u_1).$$

These are all included in the expression

$$\tfrac{1}{2}(1+D)\,u_r,$$

with the proviso that $\qquad D^m u_r = u_r.$

Similarly, the first corner of the n^{th} derived polygon is

$$2^{-n}(1+D)^n\,u_1.$$

Now let N_r^m, where r is not greater than m, be the sum of the r^{th}, $(r+m)^{\text{th}}$, $(r+2m)^{\text{th}}$, &c. coefficients of the binomial $(1+x)^n$; the above expression becomes

$$2^{-n}(N_1^m u_1 + N_2^m u_2 + \ldots + N_r^m u_r + \ldots + N_m^m u_m).$$

But, when n is infinite, its ultimate value is (as above)

$$\frac{1}{m}(u_1+u_2+\ldots+u_m).$$

Hence $\qquad L_{n=\infty}(2^{-n}N_r^m) = \dfrac{1}{m};$

and it seems remarkable that the limit is independent of r.

PROOF OF A GENERALIZATION OF NEWTON'S FORMULA FOR THE SUMS OF THE POWERS OF THE ROOTS OF AN EQUATION.

By *R. Lachlan, B.A.*, Scholar of Trinity College, Cambridge.

The present Note contains a simple proof of certain results which are due to Captain MacMahon.* The problem to be solved may be stated as follows:

"If T_m denote the sum of all the symmetric functions of weight m of the equation of the n^{th} degree

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0,$$

* See *Proceedings of the London Mathematical Society*, vol. XV.

which contain two and only two roots in each term, *e.g.* $\Sigma\alpha^p\beta^q$; to prove that

$$T_n + a_1 T_{n-1} + a_2 T_{n-2} + \ldots + a_{n-2} T_2 - \tfrac{1}{2} n(n-1) a_n = 0;$$

and to extend this theorem to the case of symmetric functions containing three, four, or any given number of the roots."

Let $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ be the roots of the equation

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n = 0,$$

so that

$$\begin{aligned}&(1-\alpha_1 x)(1-\alpha_2 x)(1-\alpha_3 x)\ldots(1-\alpha_n x)\\ &= 1 + a_1 x + a_2 x^2 + \ldots + a_n x^n\\ &\equiv f(x), \text{ say.}\end{aligned}$$

Differentiating, we get

$$\begin{aligned}f'(x) = &-\alpha_1(1-\alpha_2 x)(1-\alpha_3 x)\ldots(1-\alpha_n x)\\ &-\alpha_2(1-\alpha_1 x)(1-\alpha_3 x)\ldots(1-\alpha_n x)\\ &-\ldots\ldots\ldots\ldots\ldots\ldots,\end{aligned}$$

$$\begin{aligned}f''(x) = &\ 2\alpha_1\alpha_2(1-\alpha_3 x)(1-\alpha_4 x)\ldots(1-\alpha_n x)\\ &+2\alpha_1\alpha_3(1-\alpha_2 x)(1-\alpha_4 x)\ldots(1-\alpha_n x)\\ &+\ldots\ldots\ldots\ldots\ldots\ldots,\end{aligned}$$

and similarly,

$$\begin{aligned}f^r(x) = (-)^r r!\,\{&\ \alpha_1\alpha_2\ldots\alpha_r\ (1-\alpha_{r+1}x)\ldots(1-\alpha_n x)\\ &+\alpha_1\alpha_3\ldots\alpha_{r+1}(1-\alpha_2\ x)\ldots(1-\alpha_n x)\\ &+\ldots\ldots\ldots\ldots\ldots\ldots\}.\end{aligned}$$

Whence we get, dividing by $f(x)$,

$$\frac{(-)^r}{r!}\frac{f^r(x)}{f(x)} = \frac{\alpha_1\alpha_2\ldots\alpha_r}{(1-\alpha_1 x)\ldots(1-\alpha_r x)} + \ldots.$$

Now

$$\frac{1}{(1-\alpha_1 x)\ldots(1-\alpha_r x)} = (1+\alpha_1 x+\alpha_1^2 x^2+\ldots)(1+\alpha_2 x+\alpha_2^2 x^2+\ldots)\ldots,$$

and clearly on effecting the multiplication we shall get for the coefficient of x^m

$$\Sigma\alpha_1^{\lambda}\alpha_2^{\mu}\alpha_3^{\nu}\ldots\alpha_r^{\sigma},$$

where $\lambda+\mu+\nu+\ldots+\sigma=m$, and the summation is taken so as to include the terms found by giving all values to λ, μ, ν, &c. from 0 to m, so that $\lambda+\mu+\ldots+\sigma=m$; and if we multiply this by $\alpha_1, \alpha_2, \ldots \alpha_r$, then, we get a series of products of the form $\alpha_1^\lambda, \alpha_2^\mu, \ldots \alpha_r^\sigma$, where $\lambda, \mu, \ldots \sigma$ have all values from 1 to $m+1$, so that $\lambda+\mu+\ldots+\sigma=m+r$.

Hence if T_m denote the sum of all the symmetric functions of weight m which contain r and only r roots in each term, then we get

$$\frac{(-)^r}{r!}\frac{f^r(x)}{f(x)} = T_r + T_{r+1}x + T_{r+2}x^2 + \ldots + T_{r+m}x^m + \ldots,$$

and

$$f^r(x) = r!\,a_r + (r+1)!\,a_{r+1}x + \frac{(r+2)!}{2!}a_{r+2}x^2 + \ldots + \frac{n!}{(n-r)!}a_n x^{n-r}.$$

Hence, we have

$$\frac{(-)^r n!}{r!\,(n-r)!}a_n = \text{the coefficient of } x^{n-r} \text{ in}$$

$$\{T_r + \ldots + T_{r+m}x^m + \ldots\}\{1 + a_1x + a_2x^2 + \ldots + a_nx^n\}$$
$$= T_n + a_1T_{n-1} + a_2T_{n-2} + \ldots + a_{n-r}T_r.$$

Hence

$$T_n + a_1T_{n-1} + a_2T_{n-2} + \ldots + a_{n-r}T_r - \frac{(-)^r n!}{r!\,(n-r)!}a_n = 0.$$

Similarly, equating the coefficients of x^{m-r},

$$T_m + a_1T_{m-1} + \ldots + a_{m-r}T_r - \frac{(-)^r m!}{r!\,(m-r)!}a_m = 0.$$

EXTENSIONS OF A THEOREM OF SALMON'S.

By *Asutosh Mukhopadhyay.*

§1. "If A, B, C, D be any four points on a circle, and O any fifth point taken arbitrarily, and, if we denote by BCD the area of the triangle BCD, &c., then

$$OA^2.BCD + OC^2.ABD = OB^2.ACD + OD^2.ABC.ABC\text{''} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

This theorem is due to Dr. Salmon, (*Conics*, § 94), who arrives at it as the geometric interpretation of the analytic condition that a circle should pass through four given points. A proof from pure geometry has been supplied by Prof. Townsend, (*Geometry*, Vol. I. § 83, Cor. 3). I purpose to give here some extensions of this theorem.

§ 2. The theorem holds good when the point O is exterior to the plane containing A, B, C, D; this is clear from the analytic proof, where no restriction is laid upon the position of O; it is also rendered sufficiently obvious from Prof. Townsend's geometrical proof, which is easily seen to be literally applicable to the case in which O is not coplanar with A, B, C, D. Now, when O is outside the plane, we have the pyramid $OABCD$, which, by passing planes through O, A, C, and O, B, D, can be subdivided into the pyramids $OABD$, $OBCD$, $OABC$, $OACD$; the volumes of these minor pyramids are easily seen to be proportional to the bases on which they stand. We thus see that (i) holds, when for the triangles BCD, ABD, &c., we substitute the pyramids $OBCD$, $OABD$ &c. The theorem then can be enunciated as follows:—

"A pyramid $OABCD$ stands on a quadrangle base $ABCD$, inscribed in a circle; if the edges OA, OB, OC, OD be represented by E_1, E_2, E_3, E_4, and the volumes of the pyramids $OBCD$, $OACD$, $OABD$, $OABC$, by V_1, V_2, V_3, V_4 respectively, then,

$$V_1E_1^2 + V_3E_3^2 = V_2E_2^2 + V_4E_4^2."$$

§ 3. The theorem admits of extension, when, instead of four, we have $2n$ points on a circle. We will take as an example, the case $n=3$, for any inscribed hexagon $A_1A_2A_3A_4A_5A_6$. The hexagon can be divided up into six quadrilaterals

$$A_1A_2A_3A_4,\quad A_2A_3A_4A_5,\quad A_3A_4A_5A_6,$$
$$A_4A_5A_6A_1,\quad A_5A_6A_1A_2,\quad A_6A_1A_2A_3.$$

Now, taking any point O in the plane, and applying Salmon's Theorem to each of these quadrilaterals, we obtain six equations, the type of which is,

$$OA_1^2.A_2A_3A_4 + OA_3^2.A_1A_2A_4 = OA_2^2.A_1A_3A_4 + OA_4^2.A_1A_2A_3.$$

Adding up the six equations, and collecting the terms involving OA_1^2, its coefficient is obviously,

$$A_2A_3A_4 + A_5A_6A_2 - A_6A_2A_3 - A_4A_5A_6 = A_3A_4A_6 - A_2A_4A_5.$$

Let us now obtain the law of formation of the coefficient of any term OA_n^2; the rule is easily seen to be this:

"Arrange the natural numbers, forming the suffixes, in a cyclical order; begin to count in the direction of a watch-hand's motion from n, the suffix of A in OA_n^2; omit this one and the next; take the next two; omit the next; take the next one, and form a triangle with the three: then begin to count from the digit after n; take this one; omit the next; take the next two, and form a triangle with the three. The coefficient required is the difference of these two triangles."

We can now easily write down the identity; in the case of the hexagon ($n=3$), it is found to be,

$$OA_1^2(A_3A_4A_6 - A_2A_4A_5) + OA_3^2(A_5A_6A_2 - A_1A_4A_5) + \&c.$$
$$= OA_2^2(A_3A_5A_6 - A_4A_5A_1) + OA_4^2(A_1A_2A_5 - A_1A_3A_6) + \&c.$$

When n is greater than 3, the coefficients, instead of being differences of two triangles, become the differences of figures of a larger number of sides: the general rule is that, if the original figure is of $2n$ sides, the subsidiary figures, whose differences form the coefficients, are of $(2n-3)$ sides, n not <3. The rule for the formation of the coefficients is also extended; whereas, in the hexagon, we have to omit one, and take two next, in the case of the octagon we have to take four, and similarly for every case of increase of the number of sides by two.

It may be noticed that this theorem for $2n$ points, holds when O is anywhere in space, and thus admits of an easy generalization to pyramids as in § 2.

§ 4. A space-analogue to Salmon's Theorem is obtained with ease, by taking a sphere and five points on it. The equation of any sphere can be put in the form

$$x^2 + y^2 + z^2 + \delta + 2\lambda x + 2\mu y + 2\nu z = 0 \text{.........(ii)},$$

which may be made to satify any four conditions. If now we seek the condition that the sphere in (ii) should pass through the five given points

$$(x_1y_1z_1),\ (x_2y_2z_2),\ (x_3y_3z_3),\ (x_4y_4z_4),\ (x_5y_5z_5),$$

we have to eliminate δ, λ, μ, ν, from five equations, of which (ii) forms the type.

The condition sought therefore is

$$\begin{vmatrix} x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 \\ x_5^2+y_5^2+z_5^2 & x_5 & y_5 & z_5 \end{vmatrix} = 0$$

Now we see that $x_1^2+y_1^2+z_1^2$ represents the distance of (x_1, y_1, z_1) from $(0, 0, 0)$, and the minor which multiplies it in the expansion of the determinants in terms of the elements of the first column represents six times the volume of the tetrahedron formed by the points $(x_2y_2z_2)$, $(x_3y_3z_3)$, $(x_4y_4z_4)$, $(x_5y_5z_5)$. Hence the geometrical interpretation of the resulting condition is,

"If two tetrahedra O_1ABC, O_2ABC, standing on opposite sides of the same base ABC, be enveloped by a sphere, passing through the five angular points; then, if we take any arbitrary point O in space,

$$OA^2.BCO_1O_2+OC^2.ABO_1O_2+OO_2^2.ABCO_1$$
$$=OB_1^2.ACO_1O_2+OO_1^2.ABCO_2.$$

where BCO_1O_2, &c. represents the volumes of the tetrahedra BCO_1O_2, &c."

This theorem can be extended as in § 3, if instead of two, we have $2n$ tetrahedra; but the calculations here become very intricate. It may be noticed, that instead of the equation

$$x^2+y^2+z^2+\delta+2\lambda x+2\mu y+2\nu z=0,$$

we might as well have used the form

$$\frac{(bc)^2yz}{p'p''}+\frac{(ca)^2zx}{p''p}+\frac{(ab)^2xy}{pp'}+\text{\&c.}=0,$$

which is the equation to a sphere circumscribing a tetrahedron. cf. Salmon, *Geometry of Three Dimensions*, § 229.

April 25th, 1883.

NOTE ON THE CONNEXION BETWEEN LEGENDRE'S COEFFICIENTS AND THE COMPLETE ELLIPTIC INTEGRAL OF THE FIRST KIND.

By *G. H. Stuart, M.A.*

The equation satisfied by Legendre's coefficient is $P_n(x)$

$$(1-x^2)\frac{d^2z}{dx^2} - 2x\frac{dz}{dx} + n(n+1)z = 0 \text{..........(1)}.$$

Transform this equation by writing $\dfrac{1+k^2}{1-k^2}$ for x and $k'v$ for z, where $k' = \sqrt{(1-k^2)}$. The equation becomes

$$(1-k^2)\frac{d^2v}{dk^2} + \frac{1-3k^2}{k}\frac{dv}{dk} - v = \frac{(2n+1)^2}{1-k^2}v.$$

If $n = -\frac{1}{2}$ this becomes the differential equation satisfied by K (Cayley's *Elliptic Functions* p. 50). Therefore

$$K = \frac{C}{k'}P_{-\frac{1}{2}}\left(\frac{1+k^2}{1-k^2}\right),$$

where C is some constant.

Take the expression for $P_n(x)$ given by Todhunter, *Legendre's Functions*, p. 32, and write therein $\dfrac{1+k^2}{1-k^2}$ for x, $-\frac{1}{2}$ for n; it becomes

$$P_{-\frac{1}{2}}\left(\frac{1+k^2}{1-k^2}\right) = \frac{1}{\pi}\int_0^\pi\left(\frac{1+k^2+2k\cos\phi}{k'^2}\right)^{-\frac{1}{2}}d\phi$$

$$= \frac{k'}{\pi}\int_0^\pi \frac{d\phi}{\sqrt{(1+k^2+2k\cos\phi)}}$$

$$= \frac{2k'}{\pi(1+k)}\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{(1-\gamma^2\sin^2\theta)}},$$

$\left(\text{putting } \phi = 2\theta,\ \gamma = \dfrac{2\sqrt{k}}{1+k}\right)$

$$= \frac{2k'}{\pi}\frac{\Gamma}{1+k} = \frac{2k'K}{\pi};$$

therefore C in the above expression is equal to $\dfrac{2}{\pi}$.

Any of the expressions for $P_n(x)$ given in Todhunter's *Treatise* will give an expression for K. Thus the series on p. 11, Art. 23, gives the ordinary series for K, viz.

$$K = \tfrac{1}{2}\pi\left\{1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1.3}{2.4}\right)^2 k^4 + \dots\right\}.$$

The series (Todhunter, p. 10, Art. 19) gives

$$K = \frac{\pi}{2k'}\left\{1 - \left(\frac{1}{2}\right)^2 \frac{k^2}{k'^2} + \left(\frac{1.3}{2.4}\right)^2 \frac{k^4}{k'^4} - \dots\right\},$$

which is convergent if k is $< \sin\frac{1}{4}\pi$.

The complete solution of equation (1) is

$$z = AP_n(x) + BQ_n(x),$$

where $$Q_n(x) = CP_n(x)\int\frac{dx}{P_n^2(x)\,(x^2-1)},$$

or $$\frac{d}{dx}\frac{Q_n}{P_n} = \frac{C}{P_n^2.(x^2-1)}.$$

For $$n = -\tfrac{1}{2},\quad x = \frac{1+k^2}{1-k^2},\quad P_n(x) = \frac{2k'}{\pi}K.$$

Put $$Q_{-\frac{1}{2}}\left(\frac{1+k^2}{1-k^2}\right) = k'R,$$

and the above equation for Q_n becomes

$$\frac{d}{dk}\left(\frac{R}{K}\right) = -\frac{\frac{1}{2}\pi}{K^2kk'^2}C.$$

Comparing this with

$$\frac{d}{dk}\frac{K'}{K} = -\frac{\frac{1}{2}\pi}{K^2kk'^2},$$

we have $$\frac{d}{dk}\left(\frac{R}{K}\right) = C\frac{d}{dk}\frac{K'}{K},$$

$$\frac{R}{K} = C\frac{K'}{K} + D,$$

$$R = CK' + DK,$$

$$Q_{-\frac{1}{2}}(x) = k'(CK' + DK).$$

Therefore the complete solution of

$$(1-x^2)\frac{d^2z}{dx^2}-2x\frac{dz}{dx}-\tfrac{1}{4}z=0$$

is
$$z=A\frac{2k'}{\pi}K+Bk'(CK'+DK),$$

or
$$z=(\alpha K+\beta iK')k',$$

where α, β are independent of k.

The expression for $Q_n(x)$ involves the term $P_n(x)\log\dfrac{x+1}{x-1}$; so that the expression for K' in terms of k will involve the term $K\log k$; see the expression for $F_1(k)$ in terms of k', Cayley's *Elliptic Functions*, p. 54.

NOTE ON CAPTAIN MACMAHON'S TRANSFORMATION OF THE THEORY OE INVARIANTS.

By Professor *Sylvester*.

THE whole question as is well known consists in finding the free forms of $\Omega^{-1}0$, where

$$\Omega=a_0\delta a_1+2a_1\delta a_2+\ldots+ia_{i-1}\delta a_i;$$

but, as long ago noticed by me in the *Am: Math: Journal*, $\Omega^{-1}0$ is only a deformation of $V^{-1}0$, where

$$V=a_0\delta a_1-a_1\delta a_2+\ldots\pm a_{i-1}\delta a_i,$$

$\Omega^{-1}0$ being deducible from $V^{-1}0$ by altering the dimensions of the a elements which it contains in known numerical proportions, so that $\Omega^{-1}0$ may be said to be $V^{-1}0$ subjected to a known *strain*.*

To fix the ideas let $i=3$ and call the a's by the names a, b, c, d or, for greater simplicity, 1, b, c, d.

Let
$$b=r+s+t,$$
$$c=rs+rt+st,$$
$$d=rst.$$

* In fact the numerical multipliers of the terms in Ω may be taken perfectly arbitrary without producing any effect upon the form $\Omega^{-1}0$ than what may be represented by a *strain*.

Then the matrix

$$\frac{D(b, c, d)}{D(r, s, t)} = \begin{matrix} 1 & 1 & 1, \\ s+t & t+r & r+s, \\ st & tr & rs, \end{matrix}$$

so that

$$\frac{D(r, s, t)}{D(b, c, d)} = \begin{matrix} \frac{r^2}{(r-s)(r-t)} & \frac{s^2}{(s-r)(s-t)} & \frac{t^2}{(t-r)(t-s)}, \\ \frac{r}{(r-s)(r-t)} & \frac{s}{(s-r)(s-t)} & \frac{t}{(t-r)(t-s)}, \\ \frac{1}{(r-s)(r-t)} & \frac{1}{(s-r)(s-t)} & \frac{1}{(t-r)(t-s)}. \end{matrix}$$

Consequently

$$V = \sum \frac{r^2 - (r+s+t)\,r + (rs+rt+st)}{(r-s)(r-t)}\,\delta_r = \sum \frac{st}{(r-s)(r-t)}\,\delta_r.$$

In like manner in general for $1, a_1, a_2, \ldots a_i$ we shall find, on writing,

$$a_1 = r_1 + r_2 + \ldots + r_i,$$
$$a_2 = r_1r_2 + r_1r_3 + \ldots + r_{i-1}r_i,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$a_i = r_1r_2\ldots r_i,$$

$$V = \delta a_1 - a_1\delta a_2 + \ldots \pm a_{i-1}\delta a_i = \sum \frac{r_2r_3\ldots r_i}{(r_1-r_2)(r_1-r_3)\ldots(r_1-r_i)}\,\delta r_1.$$

Hence $\qquad V^{-1}0 = F(s_2, s_3, \ldots s_i),$

where, in general,

$$s_\omega = r_1^\omega + r_2^\omega \ldots + r_i^\omega;$$

and consequently the theory of invariants, which endoscopically treated in the ordinary way, hinges upon symmetrical functions of the differences of a set of letters is made to depend upon functions of the simple sums of powers commencing with the second power and ending with a power whose index is the order of any given finite quantic, but in the case of *perpetuants* taking in all the powers except the first.

It goes without saying that the same method applied to the *constrained* V will show that it is equal to $\Sigma\delta r_1$, so that V_0^{-1} is an arbitrary function of the differences of the r's corresponding to that hypothesis, as we know ought to be the case.

What has been established in the foregoing investigation is a principle of correspondence which its importance as a simplifying agent recalls Ivory's use of such principle in Attractions, viz. the remarkable algebraical law that any symmetrical function of the differences of a set of i quantities is a symmetrical function of the sums of the 2^{nd}, 3^{rd}, ..., i^{th} powers of another equi-numerous set.

By virtue of this principle the numerical part of the Calculus of Invariants is capable of being entirely divorced from all question of algebraical content and a Zahl-Invariant theory comes into being, in its fundamental conception, analogous to the Zahl-Geometrie of Schubert.

Further remarks on this subject will be found in the *Comptes Rendus del' Institut* presumably for March 31 and April 7 of this year.

APPROXIMATE CIRCLE-QUADRATURES.

By *A. H. Anglin, B.A., LL.B, &c.*

1. Let $ABCD$ (fig. 7) be the circumscribed square, and EF a side of inscribed square of the given circle, AC and DKB diagonals of the former. Draw a tangent to the circle at K meeting AB at G (or mark off AG equal to EF), and join GC cutting the diagonal DB in H. Then HF is very nearly equal to the side of a square equal in area to the given circle.

Taking the radius of the given circle as unity,

$$BG = 2 - \sqrt{2}, \text{ and } GC^2 = 10 - 4\sqrt{2}.$$

Since $\angle B$ of the triangle GBC is bisected by BH, we have (*Euclid*, VI. 3)

$$GH = \frac{GC.BG}{BG + BC},$$

by which it may be easily shown that

$$KH = \frac{\sqrt{2} - 1}{2\sqrt{2} - 1}.$$

Thus $$HF = \frac{3\sqrt{2} - 1}{2\sqrt{2} - 1} = \tfrac{1}{7}(11 + \sqrt{2})$$
$$= 1{\cdot}77345908\ldots.$$

The true value is 1·77245385..., so that HF is in excess of it by ·00100523... of the radius, or by rather more than the two thousandth part of the side of the square.

2. Let $ABCD$ (fig. 8) be circumscribed square, HE, EF two sides of inscribed square of given circle, AC, BD diagonals of squares. With centre E and radius EA describe a circle cutting sides of inscribed square in I and J; join IJ and produce it to meet circle at N.

Then the line GNK, drawn through N parallel to BC and terminated by the diagonals at G and K, is nearly equal to the side of a square equal in area to the given circle.

For $$NK = IE = \sqrt{2} - 1,$$

$$LN = \frac{\sqrt{(2\sqrt{2}-1)}}{\sqrt{2}}, \quad BN = \sqrt{(2\sqrt{2}-1)}.$$

Thus $$GNK = \sqrt{2} - 1 + \sqrt{(2\sqrt{2}-1)}$$

$$= 1{\cdot}76640701...,$$

which is less than the true value by ·00604684... of the radius, or by rather more than the three-hundredth part of the side of the square.

3. Let two adjacent sides of the circumscribed square touch the circle at N and H (fig. 9). Repeat the circle from centres N and H by arcs cutting sides of the inscribed square in F and E; join FE and mark off its fourth part FI on FB. Then if on the diagonal AO we mark off AK equal to IB, the line KM parallel to AH and terminated by the other diagonal at M is very nearly to the side of a square equal in area to the given circle.

We have $$EG^2 = EH^2 - HG^2,$$

from which $$EG = \frac{\sqrt{(2\sqrt{2}-1)}}{\sqrt{2}},$$

and $$FE = \sqrt{(2\sqrt{2}-1)} - 1.$$

But $$IB = \left(\frac{1}{\sqrt{2}} - \tfrac{1}{4}\right).FE$$

$$= \tfrac{1}{4}\left\{\sqrt{(2\sqrt{2}-1)} - 1\right\}(2\sqrt{2}-1) = AK.$$

Thus $$\begin{aligned} KM &= 2 - 2LK \\ &= 2 - \frac{1}{2\sqrt{2}}\{\sqrt{(2\sqrt{2}-1)} - 1\}(2\sqrt{2}-1) \\ &= 2 - \cdot 22767421\ldots \\ &= 1\cdot 77232578\ldots, \end{aligned}$$

which is less than the true value by $\cdot 0001280\ldots$ of the radius, or by nearly the *fifteen thousandth* part of the side of the square.

4. Repeat the circle as in last case by arcs AEO and AFO, and let I be the intersection of FE and the diagonal ABO (fig. 10). Join CI, and with centre C and radius CI describe a circle cutting the arc AEO in J. Then the line through J parallel to AN and terminated by the diagonals of the circumscribed square is very nearly equal to the side of a square equal in area to the given circle.

Let $CM = x$, so that side of square $= 2 - 2x$, and let $CI = b$. Then since $JN^2 + HN^2 = 1$, we get

$$(1-x)^2 + \left\{1 - \frac{1}{\sqrt{2}} + \sqrt{(b^2 - x^2)}\right\}^2 = 1,$$

a quadratic equation to find x or CM. This equation finally reduces to

$$(10 - 4\sqrt{2})\,x^2 - 4cx = 2\,(3 - 2\sqrt{2})\,b^2 - c^2,$$

where $\quad c = b^2 + \frac{3}{2} - \sqrt{2}$, and $b^2 = 1 - \dfrac{\sqrt{(2\sqrt{2}-1)}}{\sqrt{2}}$.

Solving it we find

$$CM = x = \frac{2c + (2-\sqrt{2})\sqrt{\{(10-4\sqrt{2})\,b^2 - c^2\}}}{10 - 4\sqrt{2}}$$

$$= \cdot 115904474\ldots.$$

Thus side of square $= 1\cdot 768191052\ldots$, which is less than the true value by $\cdot 004262798\ldots$ of the radius, or by less than the four-hundredth part of the side of the square.

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SUR LE DÉPLACEMENT D'UN DIÈDRE DE GRANDEUR CONSTANTE.

By *A. Mannheim.*

PRENONS une courbe plane. Sur le plan de cette courbe déplaçons un angle de grandeur constante de façon que son sommet c décrive cette courbe, tandis qu'un de ses cotés lui reste tangent.

Pour une position de cet angle, on obtient le point où l'autre coté P touche son enveloppe en projetant sur cette droite le centre de courbure de la courbe donnée correspondant à la position c du sommet de l'angle mobile.

On arrive immédiatement à ce resultat en considérant le déplacement infiniment petit de l'angle de grandeur constante, complémentaire de l'angle donné, dont un coté est la normale à la courbe issue du sommet c et dont l'autre coté est la droite P.

Voici comment on peut généraliser co résultat. Prenons une surface développable. Déplaçons un dièdre de grandeur constante de façon que son arête C coincide successivement avec les génératrices de cette surface, tandis qu'une de ses faces lui reste tangente.

Pour une position de cet angle dièdre, on obtient la droite suivant laquelle l'autre face (P) *touche son enveloppe en projetant sur cette face l'axe de courbure* de la surface développable correspondant à la position de l'arête C du dièdre mobile.*

On arrive immédiatement à ce résultat en appliquant quelques propriétés relatives au déplacement d'une figure de grandeur invariable† au déplacement infiniment petit du dièdre de grandeur constante, complémentaire du dièdre donné, dont une face est le plan normal à la surface développable mené par C et dont l'autre face est le plan (P).

Modifions l'énoncé du dernier résultat en introduisant l'arête de rebroussement A de la surface développable. Les génératrices de la surface dévéloppable sont les tangentes de A, les plans tangents de la développable sont les plans osculateurs de A et les axes de courbure de la surface développable sont les droites rectifiantes de cette courbe.

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* L'axe de courbure d'une surface développable correspondant à une génératrice est l'intersection du plan normal a la surface développable mené par cette génératrice et du plan normal mené par la génératrice inf[t] voisine.

† Voir dans mon *Cours de Géométrie descriptive* le supplément à la 24[e] Leçon.

On peut alors dire que le dièdre se déplace de façon que son arête C reste tangente à une courbe A, tandis qu'une de ses faces coincide toujours avec le plan osculateur de cette courbe.

Le résultat précédent peut alors s'énoncer ainsi:

Pour une position de cet angle, on obtient la droite suivant laquelle l'autre face (P) *touche son enveloppe en projetant sur cette face la droite rectifiante* R *correspondante à la tangente* C *de* A.

Désignons par ω l'angle constant du dièdre, par ρ le rayon de courbure de A pour le point où cette courbe est touchée par C et par r le rayon de seconde courbure de cette courbure pour le même point. La droite rectifiante R fait avec C un angle dont la tangente est égale, comme l'on sait, à $\frac{r}{\rho}$.

La droite de contact de (P) avec son enveloppe, étant la projection de R sur ce plan, fait alors avec C un angle dont la tangente est égale à $\frac{r}{\zeta}\sin\omega$. Ce dernier resultat est connu.

PRIMITIVE ROOTS OF PRIME NUMBERS AND THEIR RESIDUES.

By *A. R. Forsyth, B.A.* Fellow of Trinity College, Cambridge.

THE present paper relates to one of the most elementary and fundamental branches of the theory of numbers. The subject had been considered by many writers, the chief of whom was Legendre, before the appearance in 1801 of Gauss's *Disquisitiones Arithmeticæ*, which introduced a completely new system of notation and methods, into both the elementary and what till then had been regarded as the advanced parts; and since that date the science of discrete numbers has become perhaps the very widest in the whole range of mathematics. In the following paper I have attempted to explain the fundamental principles of an important system of theorems in a more simple and systematic manner than that in which they are presented in any of the works with which I am acquainted. Necessarily much of the ground covered is not new, and the only results that seem absolutely new are XVI–XIX; but the different propositions are logically arranged and in many cases given

in a form only implicitly contained in the treatises on the subject. The following references to the existing works on the subject will be found useful:—

Gauss, Disquisitiones Arithmeticæ, §§ 57–81.
Jacobi, Canon Arithmeticus; Introductio.
Lejeune-Dirichlet, Vorlesungen über Zahlentheorie, §§ 29–33.

Serret, Cours d'algèbre supérieure, §§ 305–321.

If p denotes a prime number and δ be the lowest index such that the congruence

$$a^{\delta} \equiv 1 \ (\text{mod.}\, p)$$

is satisfied, a being a given integer prime to p, we know that δ must be a factor of $p-1$. And the number a is said to '*belong*' or '*appertain*' to the index δ; the number of integers belonging to any index δ is $\phi(\delta)$, the number of integers less than and prime to δ.

In the case when δ is $p-1$ so that no index less than $p-1$ will allow the congruence

$$g^{p-1} \equiv 1 \ (\text{mod.}\, p)$$

to be satisfied, then g is called a *primitive root* of p; and the number of such primitive roots is $\phi(p-1)$. The congruence just written down may be replaced by

$$g^{\frac{1}{2}(p-1)} + 1 \equiv 0 \ (\text{mod.}\, p).$$

The residues of the powers $g^0, g^1, g^2, \ldots g^{p-2}$ are all different and incongruent and are therefore, in some order, the integers $1, 2, 3, \ldots p-1$; and those are the primitive roots which are the residues of the powers of g the indices of which are prime to $p-1$.

The primitive roots are obviously analogous to the proper $(p-1)^{\text{th}}$ or the *prime* $(p-1)^{\text{th}}$ roots of unity.

Any number whatever which is not a multiple of p will be congruent to some one power of g included in the preceding series; if then we have

$$c \equiv g^{\gamma} \ (\text{mod.}\, p),$$

we define γ as the index of c to the base g, or we may write it in the form

$$\text{ind}_g c \equiv \gamma \ (\text{mod.}\, p-1),$$

since

$$g^{p-1} \equiv 1 \ (\text{mod.}\, p).$$

These indices follow laws which are easily deduced.

I. The index of a product is the sum of the indices of the factors.

For if

$$a \equiv g^{\alpha} \text{ and } b \equiv g^{\beta},$$

then $ab \equiv g^{\alpha+\beta}$, or

$$\text{ind}_g (ab) \equiv \text{ind}_g a + \text{ind}_g b \, (\text{mod. } p-1).$$

Cor. $$\text{ind}_g (a^n) \equiv n \, \text{ind}_g a.$$

II. The index of a quotient (in the symbolical notation of Gauss, *D. A.* § 31) is obtained by subtracting the index of the denominator from that of the numerator.

The proof is very similar to the above.

III. The law of change from one base, a primitive root, to another, also a primitive root of the same prime modulus p, is given by

$$\text{ind}_g c \equiv \text{ind}_g h \times \text{ind}_h c \, (\text{mod. } p-1).$$

For h is the residue of some power k of g, so that

$$h \equiv g^k \, (\text{mod } p),$$

or

$$\text{ind}_g h = k.$$

Now

$$\begin{aligned} c &\equiv h^{\text{ind}_h c} \, (\text{mod. } p) \\ &\equiv g^{k \, \text{ind}_h c} (\text{mod. } p), \end{aligned}$$

or

$$\begin{aligned} \text{ind}_g c &\equiv k \, \text{ind}_h c \, (\text{mod. } p-1) \\ &\equiv \text{ind}_g h \times \text{ind}_h c \, (\text{mod. } p-1). \end{aligned}$$

Cor. Writing g in place of c we have

$$\text{ind}_g h \times \text{ind}_h g \equiv 1 \; (\text{mod. } p-1).$$

These three laws it will be noticed are exactly analogous to those which subsist between logarithms in algebra. Now in algebra there is a limitation imposed on the range from which bases may be taken for logarithms, those being most convenient which furnish positive logarithms for positive quantities greater than unity; but then negative numbers have no real logarithms. So also in the theory of congruences with any given prime modulus a limitation is imposed; the primitive roots alone can be taken as bases, for with others which are not primitive roots there will be included only some of the numbers which are not multiples of the modulus.

Negative integers do not, of course, form a separate class for, when desired, the smallest positive congruent number can be taken.

In the *Canon Arithmeticus* (Berlin, 1839) edited by Jacobi, are given tables for all prime numbers less than 1000, which contain for each prime number

(i) all the residues of some one primitive root when raised to powers $1, 2, \ldots, p-2, p-1$, these residues being, in some order, the integers $1, 2, \ldots, p-1$:

(ii) the indices of all numbers less than p when this primitive root (which varies from prime to prime) is taken for base.

It is easy to derive from these tables for any prime number all the primitive roots by taking the indices which are prime to $p-1$ and then extracting the residues which correspond to them. I have added at the end a few tables for some low primes to furnish special examples of some general propositions.

In these general propositions that follow it will be assumed that all congruences subsist with respect to a prime modulus, which may be denoted by p; that p is greater than either 2 or 3 (the propositions practically proving nothing in these cases) and is therefore odd, while $p-1$ is even; and as this will be the case except the contrary be directly stated, the ordinary method of indicating the modulus—(mod. p)—will be departed from by completely omitting specific mention of the modulus. It must be understood that no claim is made that all the propositions are new; where an old proposition is added it is for the most part because the proof is rendered easy by the method adopted. Usually the least positive index will be taken to correspond to a base, and so it can never be greater than $p-1$.

I. All the numbers less than any prime number can be arranged in pairs such that the product of a pair is congruent to unity.

For taking the residues of a primitive root g, if k be such a number, say

$$g^{\mu} \equiv k,$$

then taking $$l \equiv g^{p-1-\mu}$$

we have $$kl \equiv g^{p-1} \equiv 1.$$

(Gauss, D. A., §77, where the pairing is effected without the aid of the theory of primitive roots).

Two cases of exception arise: first, when $\mu = p - 1$ and then $k = 1$ and $l = 1$; second, when $\mu = \frac{1}{2}(p-1)$ and then $k \equiv -1$ or $p - 1$ and l is $p - 1$.

When the numbers are so arranged in pairs, the two terms of any one pair may be called conjugate.

II. If one term of such a conjugate pair be a primitive root, so is the other.

For if k be a primitive root, then μ is prime to $p - 1$, and therefore also $p - 1 - \mu$ is prime to $p - 1$, *i.e.* l is a primitive root.

Cor. 1. Hence all the primitive roots of any prime number can be arranged in conjugate pairs.

Cor. 2. Hence the product of all the primitive roots is congruent to unity; for the product of each pair is congruent to unity.

III. Wilson's Theorem easily follows from I; for arranging the numbers $2, 3, \ldots, p-2$ in pairs the product of each pair is congruent to unity; the product of 1 and $p - 1$ is congruent to -1; hence

$$1.2.3\ldots p-1 \equiv -1.$$

From this and II, *Cor.* 2, it follows that the product of all the non-primitive roots of the congruence

$$x^{p-1} \equiv 1$$

is congruent to -1.

IV. If the residues of any primitive root be, in the order of successive indices, $r_1, r_2, \ldots, r_{p-2}, 1$, then will those of the conjugate primitive root be, in the order of indices,

$$r_{p-2}, r_{p-3}, \ldots, r_2, r_1, 1.$$

For let h and h' be the roots, so that

$$hh' \equiv 1,$$

and let r_λ be any residue, so that

$$h^\lambda \equiv r_\lambda,$$

$$h'^{p-1-\mu} \equiv r_\lambda.$$

Then $$h^{\lambda}h'^{\mu} \equiv r_{\lambda}h'^{\mu} \equiv h'^{p-1} \equiv 1.$$

Now μ and λ are both $< p-1$; if they are not equal one must be the greater, say λ; then

$$h^{\lambda-\mu}(hh')^{\mu} \equiv 1;$$

$$i.e.\quad h^{\lambda-\mu} \equiv 1,$$

which is impossible (as h is a primitive root) unless $\lambda-\mu$ is zero. Hence $\lambda=\mu$, and therefore the proposition follows.

V. All the residues of any primitive root, which are not themselves primitive roots, 'belong' each to one and only one of the factors of $p-1$; let δ be such a factor. Propositions similar to the above can be inferred relating to the numbers belonging to δ; for instance, the residues of a number belonging to δ with indices prime to δ furnish all the numbers belonging to δ.

VI. Calling two indices complementary which furnish a conjugate pair (their sum is $p-1$) and taking the primitives in order as they are the residues of any primitive, then the residues of the successive primitives in order with any common index form a series which, in reversed order, gives the residues of those primitives in the same order but with the complementary index.

For if g be any primitive, then all the primitives may be expressed as

$$g,\ g^{\lambda_1},\ \ldots,\ g^{\lambda_r},\ \ldots,\ g^{p-1-\lambda_r},\ \ldots,\ g^{p-2},$$

a conjugate pair being g^{λ_r} and $g^{p-1-\lambda_r}$. Then the residues in the first case with an index c are those of

$$g^{c},\ g^{c\lambda_1},\ \ldots,\ g^{c\lambda_r},\ \ldots,\ g^{c(p-1-\lambda_r)},\ \ldots,\ g^{c(p-2)},$$

and in the second case are those of

$$g^{p-1-c},\ g^{(p-1-c)\lambda_1},\ \ldots\ g^{(p-1-c)\lambda_r},\ \ldots;$$

and in order that the proposition may be true we must have

$$g^{(p-1-c)\lambda_r} \equiv g^{c(p-1-\lambda_r)},$$

$$i.e.\qquad g^{(p-1)\lambda_r-c\lambda_r} \equiv g^{c(p-1)-c\lambda_r},$$

which is obviously satisfied as g is a primitive root of p.

VII. The residue of a primitive with any index is the conjugate of the residue of the conjugate primitive with the same index.

For if
$$\mu \equiv (g^k)^c,$$
$$\lambda \equiv (g^{p-1-k})^c,$$
then
$$\lambda\mu \equiv g^{(p-1)c} \equiv 1;$$
i. e. λ and μ are conjugate.

VIII. The same residue may arise with any the same index from a set of primitives.

For any primitive other than g being taken, as g^λ, we are to have
$$g^c \equiv g^{c\lambda},$$
or
$$c\lambda \equiv c \text{ (mod. } p-1).$$

Let δ be the G.C.M. of c and $p-1$ (the property can evidently not hold from the nature of the primitives unless δ be greater than unity), then we have
$$\lambda = 1 + t\frac{p-1}{\delta},$$

and the values that may be assigned to t are $1, 2, \ldots, \delta - 1$. As many terms of the resulting series of values for λ as are prime to $p-1$ furnish primitive roots satisfying the condition; and thus the number of times a residue could enter under the same index in a table giving the residues of all the primitive roots is the same as the number of integers prime to $p-1$ in the series
$$1 + \frac{p-1}{\delta},\ 1 + 2\frac{p-1}{\delta},\ 1 + 3\frac{p-1}{\delta},\ \ldots,\ 1 + (\delta - 1)\frac{p-1}{\delta}.$$

Cor. 1. With the same pair of primitives, if the same residue should occur with the same index, then there will be at least *two* residue-pairs.

For $0, \delta, 2\delta, \ldots, p-1$ are all values of the index c which satisfy the congruence
$$g^c \equiv g^{c\lambda};$$
and hence the number of residue-pairs is $(p-1)/\delta$.

Cor. 2. The sum of all the separate residues arising in such sets of pairs is congruent to zero (the prime number being modulus); for the sum

$$\equiv 1 + g^{\delta} + g^{2\delta} + \ldots + g^{\delta\left(\frac{p-1}{\delta}-1\right)},$$

$$\equiv \frac{g^{\delta\frac{p-1}{\delta}} - 1}{g^{\delta}},$$

$$\equiv \frac{g^{p-1} - 1}{g^{\delta} - 1} \equiv 0.$$

IX. The same residue may arise from different primitives with different indices.

Let g and g^{λ} be two primitives so that λ is prime to $p-1$; then we are to have

$$g^{k} \equiv g^{l\lambda},$$

or

$$l\lambda \equiv k \ (\text{mod. } p-1).$$

First, considering λ and k as given, a solution of this can always be found since λ is prime to $p-1$. Hence with *given* primitives the proposition follows.

Second, considering l and k as given, if δ be the G.C.M. of l and $p-1$ then δ must be a factor of k; and in that case there will be δ values of λ satisfying the congruence. Those values must then be selected which are prime to $p-1$ and they will furnish primitives as required.

Cor. If δ be unity, so that l is a primitive index, then the proposition includes that which is fundamental in the theory of primitives—that the residue of a primitive with an index prime to $p-1$ is a primitive. Moreover, having obtained the residues of g for all indices, it is easy to deduce those for g^{λ} without any further trouble than selection; for they will obviously be obtained by taking every λ^{th} term in the series of $p-1$ residues; and it was thus that the complete tables at the end were constructed.

X. With an index prime to $p-1$ no two residues of integers less than p are the same, so that in a complete table of residues under an index furnishing primitive roots all the possible residues $1, 2, \ldots, p-1$ will be found in some order.

For, taking two such integers congruent respectively to g^c and $g^{c'}$, if the residues can be the same we should have

$$g^{ck} \equiv g^{c'k},$$

or $$ck \equiv c'k \pmod{p-1}.$$

But k is prime to $p-1$; hence

$$c \equiv c' \pmod{p-1},$$

which is impossible as we assume that both c and c' are less than $p-1$.

XI. A non-primitive root of $x^{p-1} \equiv 1 \pmod{p}$ cannot, with any exponent, have as residue a primitive root.

For let the residue of g^c be a non-primitive root; then if

$$(g^c)^k \equiv h \equiv g^\lambda,$$

then $$ck \equiv \lambda \pmod{p-1}.$$

Now c and $p-1$ have a common factor, and this must therefore divide λ, *i.e.* λ is not prime to $p-1$ or h is not a primitive root.

XII. The residues of a primitive with indices differing by $\frac{1}{2}(p-1)$ have their sum congruent to zero.

For let λ and μ be indices such that

$$\lambda - \mu = \tfrac{1}{2}(p-1);$$

then $$g^{\lambda-\mu} + 1 \equiv 0 \pmod{p},$$

i.e. $$g^\lambda + g^\mu \equiv 0,$$

which proves the proposition.

It is easy to deduce as a corollary that the sum of all the terms of a period is congruent to zero.

XIII. A few tables are added at the end which will serve to furnish special examples of the above propositions; they are founded on those in the Canon Arithmeticus which are given

for a *single* primitive root of each prime number. On referring to these tables it will be found that denoting two primitive roots by g and h we have, for certain values of λ,

$$g^{\lambda} \equiv h,$$

$$h^{\lambda} \equiv g,$$

λ being of course prime to $p-1$. From these two it follows that

$$g^{\lambda^2} \equiv g,$$

or

$$\lambda^2 \equiv 1 \ (\text{mod. } p-1);$$

and the given property will occur for as many incongruent values of λ as satisfy this congruence. One value is $p-2$, the complementary of unity; in this case g and h form a conjugate pair.

XIV. Such a table can be used for several purposes.

(i) To obtain the solution of linear congruences, as indicated in Lejeune Dirichlet, *Vorlesungen über Zahlentheorie*, p. 70.

(ii) To deduce the values of n which will render possible the congruence

$$k^n \equiv l \ (\text{mod. } p),$$

k, l, p being supposed known.

Ex. $\qquad k = 95, \quad l = 73, \quad p = 181.$

We have

$$\text{ind}_2 95 = 111,$$

$$\text{ind}_2 73 = \ \ 73;$$

and the values of n are given by

$$n.111 \equiv 73 \ (\text{mod. } 180),$$

of which there is evidently no solution.

(iii) It could be used as an ordinary table of logarithms is used.

Ex. 1. Find the residue of $131 \times 81 \times 104 \times 172$ (mod. 181).

$$\begin{aligned}\text{Ind}_2\,\text{residue} &\equiv \text{ind}_2 131 + \text{ind}_2 88 + \text{ind}_2 104 + \text{ind}_2 172 \pmod{180},\\ &\equiv 43 + 65 + 167 + 22 \pmod{180},\\ &\equiv 117 \pmod{180};\end{aligned}$$

therefore

$$\text{residue} \equiv 2^{117} \pmod{181} \equiv 107.$$

Ex. 2. By what number must $131 \times 88 \times 104 \times 172$ be multiplied to have a residue 67 (mod. 181)?

If the number be x, we have, on taking indices,

$$\text{ind}_2 x + \text{ind}_2 131 + \text{ind}_2 88 + \text{ind}_2 104 + \text{ind}_2 172 \equiv \text{ind}_2 67 \pmod{180},$$

i.e. $$\text{ind}_2 x + 297 \equiv 102 \pmod{180},$$

or $$\text{ind}_2 x \equiv 165 \pmod{180};$$

therefore $$\begin{aligned} x &\equiv 2^{165} \pmod{181}\\ &= 26.\end{aligned}$$

(iv) It can be used to solve when possible (as well as to indicate the possibility of) a congruence with mod. p such as

$$x^n \equiv D$$

when n and D are given. For instance, if n be 2, then D must be the residue of some *even* power of g.

Ex. $$\begin{aligned} x^2 &\equiv 79 \pmod{181}\\ &\equiv 2^{142} \pmod{181};\end{aligned}$$

therefore $$\begin{aligned} x &\equiv 2^{71} \pmod{181}\\ &\equiv 21 \pmod{181}.\end{aligned}$$

Two obvious remarks may be here made:

First, that a primitive root cannot be a 'quadratic residue'; and second, that the quadratic residues are the residues of the even powers of any primitive root.

XV. Gauss has proved (*D. A.* §81) as follows that the sum of the primitive roots is congruent, either to zero or to ± 1, according to the form of $p-1$.

Let $a, b, c, \ldots$ be the different prime factors of $p-1$, so that

$$p-1 = a^{\alpha} b^{\beta} c^{\gamma} \ldots .$$

Then, ϕ being the ordinary functional symbol to indicate the number of integers less than and prime to that operated upon, we have

$$\phi(p-1) = \phi(a^{\alpha})\, \phi(b^{\beta})\, \phi(c^{\gamma}) \ldots,$$

and if A be a number 'belonging' to a^{α}, B to b^{β}, C to c^{γ}, ... then $ABC\ldots$ belongs to $p-1$, *i.e.* is a primitive root. Hence, if S denote the sum of the primitives,

$$S \equiv [\Sigma A].[\Sigma B].[\Sigma C]. \ldots .$$

The numbers other than A which belong to a^{α} are $A^2, A^3, \ldots, A^{a^{\alpha}-1}$ with the exception of the terms $A^a, A^{2a}, \ldots, A^{a^{\alpha}-a}$; thus

$$\Sigma A \equiv 1 + A + A^2 + \ldots + A^{a^{\alpha}-1} - [1 + A^a + A^{2a} + \ldots + A^{a^{\alpha}-a}].$$

Two cases arise:—

(i) If $\alpha = 1$, then

$$1 + \Sigma A \equiv 1 + A + A^2 + \ldots + A^{a-1} \equiv \frac{A^a - 1}{A - 1} \equiv 0,$$

i.e. $$\Sigma A \equiv -1.$$

(ii) If $\alpha > 1$, then

$$\Sigma A \equiv \frac{A^{a^{\alpha}} - 1}{A - 1} - \frac{A^{a^{\alpha}} - 1}{A^a - 1} \equiv 0,$$

and so for the other factors in $p-1$. Hence

(1) If $p-1$ be divisible by a square, there is at least one zero in the product which is equal to S, and therefore

$$S \equiv 0.$$

(2) If $p-1$ is not so divisible, then all the factors in the product are congruent to -1, and therefore we shall have

$$S \equiv \pm 1,$$

the $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$ sign being taken when the number of separate prime factors in $p-1$ is $\left\{\begin{matrix}\text{even}\\ \text{odd}\end{matrix}\right\}$.

XVI. Following this method we can find the sum of the residues of all the primitive roots with *any the same* index. Of course if this index be prime to $p-1$ then the residues are merely the primitive roots in a different order, and we have the last proposition repeated; and so we shall assume that the index is not prime to $p-1$. Denote it by c, and let S_c be the required residue; then

$$S_c \equiv [\Sigma A^c][\Sigma B^c][\Sigma C^c]\ldots(\text{mod. } p).$$

Since c is not prime to $p-1$ we may write it in the form

$$c = \mu . a^{\alpha'} . b^{\beta'} . c^{\gamma'} \ldots k^{\gamma'},$$

where μ is the product of all the factors of c prime to $p-1$.

If α' is not greater than α, then since $A^{a^\alpha} \equiv 1$ (mod. p) therefore

$$(A^{a^\alpha})^{a^{\alpha'-\alpha}} \equiv 1^{a^{\alpha'-\alpha}} \equiv 1,$$

i.e. $$A^{a^{\alpha'}} \equiv 1;$$

and therefore $$A^c \equiv 1.$$

Thus $\Sigma A^c \equiv$ number of unities equal to the number of different A's,

$$\equiv \phi(a^\alpha).$$

Similarly, if β' is less than β, we have

$$\Sigma B^c \equiv \phi(b^\beta),$$

and so for others.

If α' be $<\alpha$, let $\alpha'=\alpha-t$; then

$$\Sigma A^c \equiv 1 + A^c + A^{2c} + \ldots + A^{(a^{\alpha}-1)c}$$
$$- [1 + A^{a.c} + A^{2a.c} + \ldots + A^{(a^{\alpha}-a)c}]$$
$$\equiv \frac{A^{a^{\alpha}c}-1}{A^c-1} - \frac{A^{a^{\alpha}c}-1}{A^{ac}-1}.$$

Now, since A belongs to a^{α} while c does not contain so high a power of a, we have

$$A^{a^{\alpha}c} \equiv 1,$$

while A^c is not congruent to unity. But if t is 1, then $A^{ac} \equiv 1$; and then the value of the second fraction on the right-hand side is

$$\equiv 1 + 1 + \ldots \text{ to } a^{\alpha-1} \text{ terms}$$
$$\equiv a^{\alpha-1},$$

and so in this case when t is 1

$$\Sigma A^c \equiv -a^{\alpha-1}.$$

If t be greater than 1, then A^{ac} is not congruent to unity; and therefore

$$\Sigma A^c \equiv 0.$$

Hence we have three cases:

(1) If $\alpha' \geqq \alpha$ then $\Sigma A^c \equiv \phi(a^{\alpha})$,

(2) If $\alpha' = \alpha - 1$ ...... $\Sigma A^c \equiv -a^{\alpha-1}$,

(3) If $\alpha' < \alpha - 1$ ...... $\Sigma A^c \equiv 0$;

and so for all the prime factors in c.

If one of the numbers A, B, C, ... say M, should not belong to any power of the primes which occur in c (but should belong to the power of some prime which occurs in $p-1$) then

$\Sigma M^c \equiv \Sigma M \equiv$ 0 if M belong to a square or higher power of a prime,

$\equiv -1$ if M belongs to a linear power of a prime which occurs linearly in $p-1$.

XVII. Combining the above, we deduce the following results:—

If $p-1$ contain a prime factor squared which does not enter into c, or if it contain a prime entering into c but raised in $p-1$ to a power at least greater by 2 than in c, then the sum of the c^{th} powers of the primitive roots is congruent to zero.

If neither of these conditions be satisfied, then let $p-1$ contain λ prime factors each squared or raised to powers higher than the second, and μ prime factors each occurring linearly. By the exclusive conditions of the former result these λ factors must enter into c, any one raised to a power not less than one below the index of that one in $p-1$. If now of these λ there be λ_1 in c, such that the terms in $p-1$ being $a_1^{\alpha_1}.a_2^{\alpha_2}...a_{\lambda_1}^{\alpha_{\lambda_1}}$, then those in c are $a_1^{\alpha_1-1}.a_2^{\alpha_2-1}...a_{\lambda_1}^{\alpha_{\lambda_1}-1}$, and the remaining $\lambda-\lambda_1$ enter into c with as high as or higher powers than in $p-1$, due to these we shall have as a factor of the residue

$$(-1)^{\lambda_1}.a_1^{\alpha_1-1}.a_2^{\alpha_2-1}...a_{\lambda_1}^{\alpha_{\lambda_1}-1}\,\phi\,(a_{\lambda_1+1}^{\alpha_{\lambda_1+1}}.a_{\lambda_1+2}^{\alpha_{\lambda_1+2}}...a_{\lambda}^{\alpha_\lambda}),$$

where ϕ is the ordinary functional symbol, one property of which—$\phi\,(lm)=\phi\,(l)\times\phi\,(m)$—has been used to simplify this result. Out of the μ linear factors $b_1, b_2, ..., b_\mu$ let c contain μ' of them, either linearly or raised to any higher power, then corresponding to these we have as a factor in the residue

$$\phi\,(b_1, b_2, ..., b_{\mu'}).$$

And corresponding to the remaining $\mu-\mu'$ which do not enter into c we have $(-1)^{\mu-\mu'}$; so that as our final result we have proved that the residue of the sum of the c^{th} powers of the primitive roots is

$$\equiv(-1)^{\lambda_1+\mu-\mu'}.a_1^{\alpha_1-1}.a_2^{\alpha_2-1}...a_{\lambda_1}^{\alpha_{\lambda_1}-1}\,\phi\,(a_{\lambda_1+1}^{\alpha_{\lambda_1+1}}.a_{\lambda_1+2}^{\alpha_{\lambda_1+2}}...a_{\lambda}^{\alpha_\lambda}b_1b_2...b_{\mu'}) \pmod{p},$$

while it will be remembered from the various statements about the value of c that

$$c=Ta_1^{\alpha_1-1}.a_2^{\alpha_2-1}...a_{\lambda_1}^{\alpha_{\lambda_1}-1}.a_{\lambda_1+1}^{\alpha_{\lambda_1+1}+t_1}.a_{\lambda_1+2}^{\alpha_{\lambda_1+2}+t_2}...a_{\lambda}^{\alpha_\lambda+t_\lambda}b_1^{s_1}b_2^{s_2}...b_{\mu'}^{s_{\mu'}},$$

in which $t_1, t_2, \ldots, t_\lambda$ may be zero or positive integers, $s_1, s_2, \ldots, s_{\mu'}$ may be unity or any greater positive integers, and T is the product of the prime factors of c which do not enter into $p-1$, while μ is the number of prime factors entering linearly into $p-1$.

XVIII. Consider now some examples.

Ex. 1. Let $p=37$, so that $p-1=2^2.3^2$; and therefore μ and μ' are both zero.

Let $c=2$, then $p-1$ contains a factor 3^2 not entering into c; and therefore

$$S_2 \equiv 0.$$

Similarly $S_4, S_8, S_{10}, S_{14}, S_{16}, S_{20}, S_{22}, S_{26}, S_{28}, S_{32}, S_{34}$; $S_3, S_9, S_{15}, S_{21}, S_{27}, S_{33}$ are all congruent to zero.

Again $S_1, S_5, S_7, S_{11}, S_{13}, S_{17}, S_{19}, S_{23}, S_{25}, S_{29}, S_{31}, S_{35}$ are all congruent to zero; for the indices are prime to 36 and therefore only furnish the primitives in different orders. And the sum of the primitives is congruent to zero by Gauss's theorem because $p-1$ contains a square factor.

The only sums remaining to consider are $S_6, S_{12}, S_{18}, S_{24}, S_{30}$.

When $c=6=2.3$, $\lambda_1=2=\lambda$, $\alpha_1=2=\alpha_2$, then

$$S_6 \equiv 6.$$

Similarly $$S_{30} \equiv 6.$$

So for $c=12=2^2.3$, $\lambda_1=1$, $\lambda=2$, $\alpha_1=2=\alpha_2$, $\alpha_1=3$, $\alpha_2=2$; and

$$S_{12} \equiv -3\phi(2^2) \equiv -6.$$

Similarly $$S_{24} \equiv -6.$$

Let $c=18=3^2.2$; in the same way

$$S_{18} = -2\phi(3^2) = -12.$$

Ex. 2. Let $p=31$, so that $p-1=2.3.5$.

In this case $$S_c \equiv (-1)^{3-\mu'}\phi(b_1b_2\ldots b_{\mu'}),$$

in which $b_1=2$, $b_2=3$, $b_3=5$.

(α) $c=14=2.7$; here $\mu'=1$, corresponding to factor 2; then

$$S_{14} \equiv \phi(2) \equiv 1.$$

(β) $c = 24 = 2^3.3$; here $\mu' = 2$, corresponding to factors 2 and 3; thus

$$S_{24} \equiv -\phi(2.3) \equiv -2;$$

and so for others, all the results agreeing with the tables annexed.

Ex. 3. Let $p = 631$, so that $p - 1 = 2.3^2.5.7$; to find S_{300}.
Here $c = 300 = 3.2^2.5^2$; and we have $\lambda = 1$, $\lambda_1 = 1$; $\mu = 3$, $\mu' = 2$; and therefore

$$S_{300} \equiv 3\phi(2.5) \equiv 12.$$

XIX. When the sum of the residues of the c^{th} powers of the primitive roots of p is $\pm\mu$, these residues are really μ repetitions of one set. It is evident from the form of μ that it is a factor of $\phi(p-1)$; and therefore in this case there would be

$$\frac{1}{\mu}\phi(p-1)$$

such sets.

Suppose for the moment that the A's in the formation of S_c are the only terms that cause the residue to differ from unity; then

$$\begin{aligned} S_c &\equiv [\Sigma A^c][\Sigma B^c][\Sigma C^c]\ldots \\ &\equiv A^c[\Sigma B^c][\Sigma C^c]\ldots \\ &\quad + A^{2c}[\Sigma B^c][\Sigma C^c]\ldots \\ &\quad\quad + A^{3b}[\Sigma B^c][\Sigma C^c]\ldots \\ &\quad\quad\quad + \ldots\ldots\ldots\ldots\ldots\ldots. \end{aligned}$$

Now by the supposition $A^c \equiv A^{2c} \equiv A^{3c} \equiv \ldots \equiv 1$ and so each line furnishes the same quota towards the residue; but each line consists of the residues from the different primitives; and so these may be divided into classes. The step from this to the proposition as enunciated is easy.

XX. Several of the elementary properties of the quadratic and non-quadratic residues of a prime number p can be deduced by means of the primitive roots.

It has already been remarked that the quadratic residues are those of the even powers of a primitive root, while the

non-quadratic residues are those of the odd powers. From these two results there follow at once the propositions:

(1) that the product of two quadratic residues, or of two non-quadratic residues, is a quadratic residue,

(2) that the product of a quadratic residue and a non-quadratic residue is a non-quadratic residue,

and by continued application of these two the character of the product of any number of residues can be ascertained; for all that has to be done is to find whether the sum of the indices of the component residues is odd or even.

Of the $\frac{1}{2}(p-1)$ non-quadratic residues $\phi(p-1)$ are furnished by the primitive roots; it is not difficult to prove that any integer belonging to a factor of $p-1$ which contains all the powers of 2 occurring in $p-1$ is a non-quadratic residue and that the number of these arising from such factors exactly makes up, with the $\phi(p-1)$, the whole of the non-quadratic residues.

XXI. Let α be a quadratic residue of a prime p, g a primitive root; then

$$g^{\nu} \equiv \alpha \ (\text{mod. } p).$$

Let μ be the index to base g of $p-\alpha$, so that

$$g^{\mu} \equiv p-\alpha \ (\text{mod. } p);$$

and therefore $$g^{\mu}+g^{\nu} \equiv 0 \ (\text{mod. } p),$$

i.e. $$\mu \equiv 2l+\tfrac{1}{2}(p-1) \ (\text{mod. } p-1).$$

If then p be of the form $4n+1$, μ is even and therefore $p-\alpha$ is a quadratic residue; but if p be of the form $4n+3$ then μ is odd, and therefore $p-\alpha$ is a non-quadratic residue. Thus for prime numbers p of the form $4n+1$ the numbers α and $p-\alpha$ are both quadratic or both non-quadratic residues; while for prime numbers p of the form $4n+3$ the numbers α and $p-\alpha$ are one a quadratic residue and the other a non-quadratic residue.

These results can be derived from the congruence

$$(p-\alpha)^{\frac{1}{2}(p-1)} \equiv (-1)^{\frac{1}{2}(p-1)}\alpha^{\frac{1}{2}(p-1)} \ (\text{mod. } p),$$

which can be at once verified by the expansion of the binomial on the left-hand side.

Tables.

The numbers in the columns headed by asterisks are the primitive roots.

13.

Indices...	Prime Roots.	*1	2	3	4	*5	6	*7	8	9	10	*11	12
	2	2	4	8	3	6	12	11	9	5	10	7	1
	6	6	10	8	9	2	12	7	3	5	4	11	1
	11	11	4	5	3	7	12	2	9	8	10	6	1
	7	7	10	5	9	11	12	6	3	8	4	2	1

17.

dices...	Prime Roots.	*1	2	*3	4	*5	6	*7	8	*9	10	*11	12	*13	14	*15	16
	3	3	9	10	13	5	15	11	16	14	8	7	4	12	2	6	1
	10	10	15	14	4	6	9	5	16	7	2	3	13	11	8	12	1
	5	5	8	6	13	14	2	10	16	12	9	11	4	3	15	7	1
	11	11	2	5	4	10	8	3	16	6	15	12	13	7	9	14	1
	14	14	9	7	13	12	15	6	16	3	8	10	4	5	2	11	1
	7	7	15	3	4	11	9	12	16	10	2	14	13	6	8	5	1
	12	12	8	11	13	3	2	7	16	5	9	6	4	14	15	10	1
	6	6	2	12	4	7	8	14	16	11	15	5	13	10	9	3	1

19.

...	Prime Roots.	*1	2	3	4	*5	6	*7	8	9	10	*11	12	*13	14	15	16	*17	18
	2	2	4	8	16	13	7	14	9	18	17	15	11	3	6	12	5	10	1
	13	13	17	12	4	14	11	10	16	18	6	2	7	15	5	8	9	3	1
	14	14	6	8	17	10	7	3	4	18	5	13	11	2	9	12	16	15	1
	15	15	16	12	9	2	11	13	5	18	4	3	7	10	17	8	6	14	1
	3	3	9	8	5	15.	7	2	6	18	16	10	11	14	4	12	17	13	1
	10	10	5	12	6	3	11	15	17	18	9	14	7	13	16	8	4	2	1

23.

Indices... Prime Roots.	*1	2	*3	4	*5	6	*7	8	*9	10	11	12	*13	14	*15	16	*17	18	*19	20	*21	22
5	5	2	10	4	20	8	17	16	11	9	22	18	21	13	19	3	15	6	7	12	14	1
10	10	8	11	18	19	6	14	2	20	16	22	13	15	12	5	4	17	9	21	3	7	1
20	20	9	19	12	10	16	21	6	5	8	22	3	14	4	11	13	7	2	17	18	15	1
17	17	13	14	8	21	12	20	18	7	4	22	6	10	9	15	2	11	3	5	16	19	1
11	11	6	20	13	5	9	7	8	19	2	22	12	17	3	10	18	14	16	15	4	21	1
21	21	4	15	16	14	18	10	3	17	12	22	2	19	8	7	9	5	13	20	6	11	1
19	19	16	5	3	11	2	15	9	10	6	22	4	7	18	20	12	21	8	14	13	17	1
15	15	18	17	2	7	13	11	4	14	3	22	8	5	6	21	16	10	12	19	9	20	1
7	7	3	21	9	17	4	5	12	15	13	22	16	20	2	14	6	19	18	11	8	10	1
11	14	12	7	6	15	3	19	13	21	8	22	9	11	16	17	8	20	4	10	2	5	1

29.

Prime Roots.	*1	2	*3	4	*5	6	7	8	*9	10	*11	12	*13	14	*15	16	*17	18	*19	20	21	22	*23	24	*25	26	*27	28
2	2	4	8	16	3	6	12	24	19	9	18	7	14	28	27	25	21	13	26	23	17	5	10	20	11	22	15	1
8	8	6	19	7	27	13	17	20	15	4	3	24	18	28	21	20	10	22	2	16	12	9	14	25	26	5	11	1
3	3	9	27	23	11	4	12	7	21	5	15	16	19	28	26	23	2	6	18	25	17	22	8	24	14	13	10	1
19	19	13	15	24	21	22	12	25	11	6	27	20	3	28	10	16	14	5	8	7	17	4	18	23	2	9	26	1
18	18	5	3	25	15	9	17	16	27	22	19	23	8	28	11	24	26	4	14	20	12	13	2	7	10	6	21	1
14	14	22	18	20	19	5	12	23	3	13	8	25	2	28	15	7	11	9	10	24	17	6	26	16	21	4	27	1
27	27	4	21	16	26	6	17	24	10	9	11	7	15	28	2	25	8	13	3	23	12	5	19	20	18	22	14	1
21	21	6	10	7	2	13	12	20	14	4	26	24	10	28	8	23	19	22	27	16	17	9	15	25	3	5	18	1
26	26	9	2	23	18	4	17	7	8	5	14	16	10	28	3	20	27	6	11	25	12	22	21	24	15	13	19	1
10	10	13	14	24	8	22	17	25	18	6	2	20	26	28	19	16	15	5	21	7	12	4	11	23	27	9	3	1
11	11	5	26	25	14	9	12	16	2	22	10	23	21	28	18	24	3	4	15	20	17	13	27	7	19	6	8	1
15	15	22	11	20	10	5	17	23	26	13	21	25	27	28	14	7	18	9	19	24	12	6	3	16	8	4	2	1

Indices...

31.

Prime Roots. / Indices	*1	2	3	4	5	6	*7	8	9	10	*11	12	*13	14	15	16	*17	18	*19	20	21	22	*23	24	25	26	27	28	*29	30
3	3	9	27	19	26	16	17	20	29	25	13	8	24	10	30	28	22	4	12	5	15	14	11	2	6	18	23	7	21	1
17	17	10	15	7	26	8	12	18	27	25	22	2	3	20	30	14	21	16	24	5	23	19	13	4	6	9	29	28	11	1
13	13	14	27	10	6	16	22	7	29	5	3	8	11	19	30	18	17	4	21	25	15	9	24	2	26	28	23	20	12	1
24	24	18	29	14	26	4	3	10	23	25	11	16	12	9	30	7	13	2	17	5	27	28	21	8	6	20	15	19	22	1
22	22	19	15	20	6	8	21	28	27	5	17	2	13	7	30	9	21	16	11	25	23	10	3	4	26	14	19	18	24	1
12	12	20	23	28	26	2	24	9	15	25	21	4	17	18	30	19	11	8	3	5	29	7	22	16	6	10	27	14	13	1
11	11	28	29	9	6	4	13	19	23	5	24	16	21	14	30	20	3	2	22	25	3	18	12	8	26	7	15	10	17	1
21	21	7	23	18	6	2	11	14	15	5	12	4	22	28	30	10	24	8	13	25	29	20	17	16	26	19	27	9	3	1

The following tables are similar to those in the Canon Arithmeticus, the only difference being that a different primitive root is taken for base. In each case, the first table gives the residues of the powers, the second gives the indices. The first table shews that, if $p=41$, then

$$7^1 \equiv 7, \quad 7^{15} \equiv 14, \quad 7^{28} \equiv 4, \quad 7^{22} \equiv 33, \text{ \&c.},$$

and the second table shews that, in the case of the same prime,

$$\text{ind}_7\, 5 \equiv 18 \text{ (mod. 40)}, \quad \text{ind}_7 18 \equiv 24 \text{ (mod. 40)}, \quad \text{ind}_7 21 \equiv 26 \text{ (mod. 40)}, \quad \text{ind}_7 32 \equiv 30 \text{ (mod. 40)}.$$

Were it desirable to have some other primitive as base, the transformation from h to g could be effected by the formula $\text{ind}_g c \equiv \text{ind}_g h \times \text{ind}_h c$ (mod. $p-1$).

Numbers above which an asterisk is placed are primitive roots.

41.

Residues.

	0	1	2	3	4	5	6	7	8	9
		*7	8	*15	23	38	20	*17	37	*13
1	9	*22	31	*12	2	14	16	*30	5	*35
2	40	*34	33	*26	18	3	21	*24	4	*28
3	32	*19	10	*29	39	27	25	*11	36	*6
4	1									

Indices.

	0	1	2	3	4	5	6	7	8	9
		40	14	25	28	18	39	1	2	10
1	32	37	13	9	15	3	16	7	24	31
2	6	26	11	4	27	36	23	35	29	33
3	17	12	30	22	21	19	38	8	5	34
4	20									

43.

	0	1	2	3	4	5	6	7	8	9
		*3	9	27	38	*28	41	37	25	32
1	10	*30	4	*12	36	22	23	*26	35	*19
2	14	42	40	*34	16	*5	15	2	6	*18
3	11	*33	13	39	31	7	21	*20	17	8
4	24	*29	1							

	0	1	2	3	4	5	6	7	8	9
		42	27	1	12	25	28	35	39	2
1	10	30	13	32	20	26	24	38	29	19
2	37	36	15	16	40	8	17	3	5	41
3	11	34	9	31	23	18	14	7	4	33
4	22	6	21							

47.

Residues.

	0	1	2	3	4	5	6	7	8	9
		*5	25	*31	14	*23	21	*11	8	*40
1	12	*13	18	*43	27	*41	17	*38	2	*10
2	3	*15	28	46	42	*22	16	*33	24	*26
3	36	*39	7	*35	34	*29	4	*20	6	*30
4	9	*45	37	*44	32	*19	1			

Indices.

	0	1	2	3	4	5	6	7	8	9
		46	18	20	36	1	38	32	8	40
1	19	7	10	13	4	21	26	16	12	45
2	37	6	25	5	28	2	29	14	22	35
3	39	3	44	27	34	33	30	42	17	31
4	9	15	24	13	43	41	23			

53.

	0	1	2	3	4	5	6	7	8	9
		*2	4	*8	16	*32	11	*22	44	*35
1	17	*34	15	30	7	*14	28	*3	6	*12
2	24	*48	43	*33	13	*26	52	*51	49	*45
3	37	*21	42	*31	9	*18	36	*19	38	23
4	46	*39	25	*50	47	*41	29	*5	10	*20
5	40	*27	1							

	0	1	2	3	4	5	6	7	8	9
		52	1	17	2	47	18	14	3	34
1	48	6	19	24	15	12	4	10	35	37
2	49	31	7	39	20	42	25	51	16	46
3	13	33	5	23	11	9	36	30	38	41
4	50	45	32	22	8	29	40	44	21	28
5	43	27	26							

END OF VOL. XIII.

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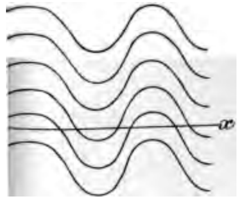
2.



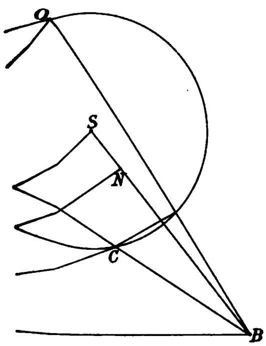


Fig. 6.

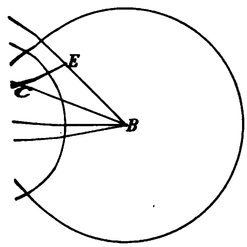
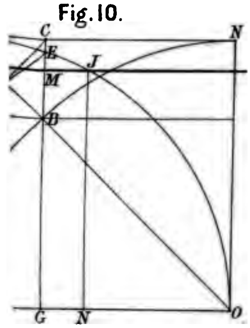


Fig. 10.



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